FERNANDO E. SERRANO*, JOSEP M. ROSSELL**

COMPLETE KINEMATIC ANALYSIS OF THE STEWART–GOUGH PLATFORM BY UNIT QUATERNIONS

ABSTRACT

In this paper, a complete analysis of Stewart–Gough platform kinematics by unit quaternions is proposed. Even when unit quaternions have been implemented in different applications (including a kinematic analysis of the Stewart platform mechanism), the research regarding the application of this approach is limited only to the analysis of some issues related to the kinematic properties of this parallel mechanism. For this reason, a complete analysis of the Stewart–Gough platform is shown.

The derivation of the inverse and forward kinematics of the Stewart platform using unit quaternions shows that they are suitable to represent the orientation of the upper platform due to their simplicity, equivalence, and compact representation as compared to rotation matrices. Then, the leg velocities are derived to compute these values under different conditions.

Keywords: *Stewart platform, parallel kinematics, unit quaternions, robotics*

ANALIZA KINEMATYCZNA PLATFORMY STEWARTA–GOUGHA Z ZASTOSOWANIEM KWATERNIONÓW

W niniejszym artykule zaproponowano analizę kinematyki platformy Stewarta–Gougha z zastosowaniem kwaternionów. Mimo że kwaterniony znalazły zastosowanie w różnych aplikacjach (w tym w analizie kinematycznej mechanizmu platformy Stewarta), to ich zastosowanie ogranicza się jedynie do analizy własności kinematycznych mechanizmów równoległych. Z tego powodu przedstawiono pełną analizę kinematyczną platformy Stewarta–Gougha. Uzyskanie kinematyki prostej i odwrotnej platformy Stewarta z zastosowaniem kwaternionów pokazuje, że są one odpowiednie do reprezentowania orientacji górnej platformy. Przede wszystkim cechują się prostotą oraz zwartą reprezentacją w porównaniu do macierzy obrotów. Następnie wyznacza się prędkości podpór, w celu obliczenia wartości w różnych warunkach.

Słowa kluczowe: *platforma Stewarta, kinematyka równoległa, kwaternion, robotyka*

1. INTRODUCTION

The Stewart–Gough platform is a well- known parallel mechanism that consists of a lower (fixed) platform and, upper (moving) platform that are connected by six legs or limbs. This mechanism is mainly used in different kinds of applications such as aircraft, vehicle simulators and other implementations in the industrial and bio-mechanical fields (Nanua, Waldron 1989; Tu *et al.* 2004; Chen *et al.* 2011; Omran, Kassem 2011; Yang *et al.* 2011; Morell *et al.* 2013). It is important to obtain an accurate and well defined kinematic model to analyze different issues regarding the orientation of the Stewart–Gough platform, in order to study the inverse and forward kinematics. There are some kinematic and singularity analysis considering the orientation of the upper platform implementing some novelties such as the classic kinematics models that are used in the kinematic analysis of serial and parallel robots and other applications (Funda, Paul 1990; Su *et al.* 2002; Duindam, Stramigioli 2008; Cao *et al.* 2010; He *et al.* 2010; Morell *et al.* 2012; Portman *et al.* 2012; Tari *et al.* 2012; Zhang 2012; Chen, Fu 2013; Quoc, Thanh 2013; Lou *et al.* 2014). Other derivations of the forward and inverse kinematics can be found in (Huang, Fu 2004; Huang, Fu 2005; Chen, Fu 2006; Ghobakhloo *et al.* 2006; Dongya *et al.* 2007; Chen, Fu 2008), where a dynamic model of the Stewart platform is obtained based on the forward kinematics of the model in order to design efficient control strategies for this mechanism. In (Wang *et al.* 2011) a new forward kinematic methodology is derived using the independent components analysis together with the Nelder and Mead algorithm, where an efficient forward kinematic study is done to obtain the orientation of the platform in the space work. Despite this fact, the development of the kinematic analysis models for the Stewart–Gough platform is limited only to forward and inverse kinematics.

In this work, an extension of previous studies to obtain a complete kinematic analysis of the Stewart–Gough platform by unit quaternions is proposed to show other kinematic properties such as angular and linear velocity of the platform and legs velocities. The Stewart–Gough kinematics is obtained from unit quaternions, so the problem to

* Universidad Tecnológica Centroamericana (UNITEC) Tegucigalpa, Honduras; e-mail: serranofer@eclipso.eu
** Universitat Politecnica de Catalunya, Department of Mathematics, Spain; e-mail: josep.maria.rossell@upc.edu

be solved here is to obtain a simplified mathematical representation of the forward and inverse kinematics of this parallel mechanism from unit quaternions considering that other mathematical representations, such as rotation matrices among others, are very complex.

It is important to consider that a goal of this study is to obtain the inverse and forward kinematic model for this mechanism. Then based on these results, the angular and linear velocities are obtained to extend these outcomes.

As a basis for this work, it is important to consider that only the rotation angles of the Stewart platform are considered starting from the fact that only the rotation matrix based on Euler angles is implemented, taking into account that a rotational and translational movement can be represented by pure rotations.

Apart from the previous contribution, the main objective of this article is to extend this work to obtain a feasible dynamic model, based on Euler–Lagrange equations, to derive efficient control strategies in order to stabilize this parallel mechanism in applications such as machine-tools and, flight / vehicle simulators.

Nomenclature and abbreviations

It is important to notice that all of the quaternion operations used in this study are explained in (Chou 1992), and the reader can refer to this reference for more details. To define a rotation matrix using unit quaternions that represent Euler angles, consider the following quaternion $p = [p_0, p_1, p_2, p_3]^T$ where p_0 and $\vec{p} = [p_1, p_2, p_3]$ are the scalar and vector quaternion part, respectively. The unitary quaternion can be represented by $p = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)u$ with $\cos\left(\frac{\theta}{2}\right) = p_0$ and $\sin\left(\frac{\theta}{2}\right) = \|\vec{p}\|$, with the unitary vector $u = \frac{\vec{p}}{\|\vec{p}\|}$. The base platform frame (fixed platform) is denoted by (x, y, z) while the top platform frame (moving platform) is denoted by (x', y', z'). The vector from the origin of the base frame to the top frame is denoted by P. The vector from the base frame (x, y, z) to the bottom frame (x_i, y_i, z_i) of each leg is represented by B_i for $i = 1, \ldots, 6$, while the vector from the top frame (x', y', z') to the top frame (x_j, y_j, z_j) of each leg is represented by A_j for $j = 1, \ldots, 6$. For this purpose it is important to notice the difference between the lowercase p and uppercase P, where in the first case p is a unit quaternion and P is the vector from the origin (x, y, z) to (x', y', z'). $\otimes$ is the quaternion product of the previously defined quaternions and $+$ is the quaternion addition. Matrix $R(p)$ is the rotation matrix obtained by the unit quaternions, L_i is the leg vector, L_{cg} is the vector V_{cg} with respect to the base frame (x, y, z) and Vel_{cg} is the linear velocity vector of the Stewart platform. The vector from the top frame (x', y', z') to the center of gravity is called V_{cg} whose components are V_{cgx}, V_{cgy} and V_{cgz} (see Fig. 1) These vectors are necessary for the derivation of the forward and inverse kinematics of this mechanism due to the simplicity of quaternions in comparison with other approaches as shown in (Nanua, Waldron 1989; Choi *et al.* 2007; Wang *et al.* 2011).

2. STEWART PLATFORM KINEMATICS BY UNIT QUATERNIONS

The forward kinematics consists of finding the orientation of the Stewart–Gough platform given specific leg lengths. Before deriving the respective equations for the inverse and forward kinematics, a representation of a rotation matrix by unit quaternions is derived in order to represent the orientation by Euler angles as shown in (1), (2) and (3).

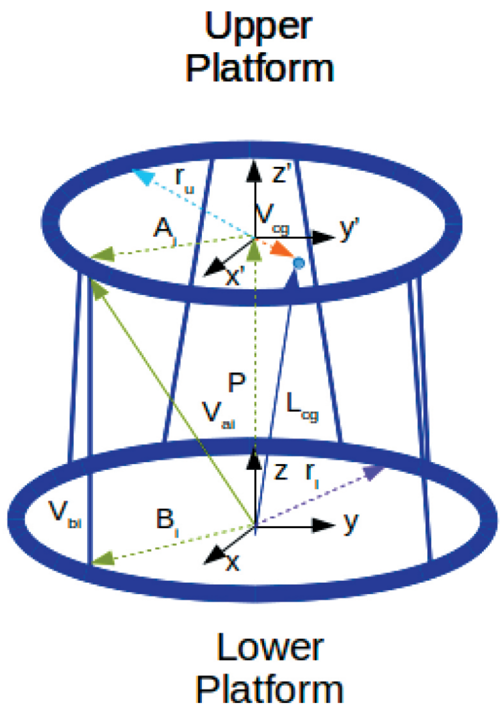

Fig. 1. Stewart platform diagram

In Figure 1 the frame axes and vectors of the Stewart platform are depicted. The Euler angles are represented in unit quaternions by the following operations (Campa *et al.* 2006; Fresk, Nikolakopoulos 2013):

$$R_x(p) = p \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \otimes p^* = \begin{bmatrix} 0 \\ p_0^2 + p_1^2 - p_2^2 - p_3^2 \\ 2(p_1 p_2 + p_0 p_3) \\ 2(p_1 p_2 - p_0 p_2) \end{bmatrix} \quad (1)$$

$$R_y(p) = p \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes p^* = \begin{bmatrix} 0 \\ 2(p_1 p_2 - p_0 p_3) \\ p_0^2 - p_1^2 + p_2^2 - p_3^2 \\ 2(p_2 p_3 + p_0 p_1) \end{bmatrix} \tag{2}$$

$$R_z(p) = p \otimes \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes p^* = \begin{bmatrix} 0 \\ 2(p_1 p_3 + p_0 p_2) \\ 2(p_2 p_3 - p_0 p_1) \\ p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix} \tag{3}$$

Considering that $R_x(p)$, $R_y(p)$ and $R_z(p)$ are vector quaternions, this means that their scalar part is zero, the following representation of the rotation matrix is obtained as a 3 × 3 matrix because these are vector quaternions:

$$R(p) = [R_x(p)\ R_y(p)\ R_z(p)] \tag{4}$$

where the Euler angles are obtained as follows (Fresk, Nikolakopoulos 2013):

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \arctan2\left(2(p_0 p_1 + p_2 p_3), p_0^2 - p_1^2 - p_2^2 + p_3^2\right) \\ \arcsin\left(2(p_0 p_2 - p_3 p_1)\right) \\ \arctan2\left(2(p_0 p_3 + p_1 p_2), p_0^2 + p_1^2 - p_2^2 - p_3^2\right) \end{bmatrix} \tag{5}$$

A rotation matrix obtained from quaternions can also be derived by the following formulas (Campa *et al.* 2006):

$$\begin{aligned} R(p) &= \left(\eta^2 - \varepsilon^T \varepsilon\right) I + 2\eta S(\varepsilon) + 2\varepsilon\varepsilon^T = \\ &= \left(\eta^2 + \varepsilon^T \varepsilon\right) I + 2\eta S(\varepsilon) - 2S(\varepsilon)^T S(\varepsilon) \end{aligned} \tag{6}$$

where:

$$\eta = p_0 \varepsilon = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{7}$$

where I is the identity matrix. Finally:

$$S(\varepsilon) = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \tag{8}$$

2.1. Inverse kinematics of Stewart platform by unit quaternions

As explained in (Liu *et al.* 2000; Ji, Wu 2001; Wang *et al.* 2011) the inverse kinematics of the Stewart–Gough platform is obtained straightforwardly by computing the leg lengths for a given orientation represented by unit quaternions. So, in order to obtain the inverse kinematic model, the origin of the base and top axes of the legs are positioned in the following coordinates:

$$b_i = r_l \left(\cos\left(\frac{2i-2}{6}\right)\pi, \sin\left(\frac{2i-2}{6}\right)\pi, 0 \right) \tag{9}$$

for $i = 1, 3, 5$

$$b_i = r_l \left(\cos\left(\frac{2i-3}{6}\right)\pi, \sin\left(\frac{2i-3}{6}\right)\pi, 0 \right) \tag{10}$$

for $i = 2, 4, 6$

$$a_i = r_u \left(\cos\left(\frac{2i-3}{6}\right)\pi, \sin\left(\frac{2i-3}{6}\right)\pi, 0 \right) \tag{11}$$

for $i = 1, 3, 5$

$$a_i = r_u \left(\cos\left(\frac{2i-3}{6}\right)\pi, \sin\left(\frac{2i-3}{6}\right)\pi, 0 \right) \tag{12}$$

for $i = 2, 4, 6$

where r_l is the radius of the base platform and r_u the radius of the top platform. Coordinates b_i and a_i represent the origin of the legs axes in the base and top platform with respect to the (x, y, z) and (x', y', z') axes, respectively.

With the previous definitions, the inverse kinematics of the Stewart platform with unit quaternions can be obtained by the following (Ji, Wu 2001):

$$\begin{aligned} L_i = &\left(p \otimes d_x \otimes p^*\right) a_{ix} + \\ &+ \left(p \otimes d_y \otimes p^*\right) a_{iy} + \\ &+ \left(p \otimes d_z \otimes p^*\right) a_{iz} + P - B_i \end{aligned} \tag{13}$$

Considering that $p \otimes d_r \otimes p^*$ is a vector quaternion for $r = x, y, z$, then vectors A_i and B_i (as shown in Fig. 1) are represented by the following:

$$A_i = \begin{bmatrix} a_{ix} \\ a_{iy} \\ a_{iz} \end{bmatrix}, \quad B_i = \begin{bmatrix} b_{ix} \\ b_{iy} \\ b_{iz} \end{bmatrix} \tag{14}$$

Defining the following vector quaternions:

$$d_x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad d_y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad d_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{15}$$

vector quaternion P is given by:

$$P = \left(p \otimes d_z \otimes p^*\right) r \tag{16}$$

where r is the distance between the origins of the (x, y, z) and (x', y', z') axes when these two axes are aligned in respect to the frame. Then, the leg vector obtained in (13) can be calculated using a matrix representation of the rotation matrix as shown in (17):

$$L_i = R(p) A_i + P - B_i \tag{17}$$

and finally the leg lengths given a specific orientation defined by $R(p)$ is obtained by (Morell *et al.* 2013):

$$L_i^2 = \left\| R(p) A_i + P - B_i \right\| \tag{18}$$

From (18) the leg lengths can be calculated for any orientation given by the rotation matrix $R(p)$ in terms of quaternions.

2.2. Forward kinematics of Stewart platform by unit quaternions

The main idea, in order to find the orientation of the platform given the legs lengths, is to obtain a nonlinear algebraic equations system to be solved numerically. To derive the forward kinematics of the Stewart platform by this approach, the following assumption must be considered (Ji, Wu 2001):

Assumption 1: *Since the two hexagons of the base platform and top platform are similar, the following condition holds:*

$$A_i = \mu B_i \tag{19}$$

for $i = 1, \ldots, 6$. *The constant* μ *is called the scaling factor.*

This assumption is important because the number of variables to be solved from the obtained system of nonlinear algebraic equations is reduced, as shown in the Appendix. To establish the system of nonlinear algebraic equations to find a unique solution it is necessary to express (18) in the following form:

$$L_i^2 = \left(R(p) A_i + P - B_i\right)^T \left(R(p) A_i + P - B_i\right) \tag{20}$$

for $i = 1, \ldots, 6$, where P is the vector quaternion specified in (16). The coordinates of the leg B_i with respect to the base frame (x, y, z) are:

$$B_i = \begin{bmatrix} b_{ix} \\ b_{iy} \\ 0 \end{bmatrix} \tag{21}$$

and the coordinates of the leg A_i with respect to the top frame (x', y', z') are denoted by (22). Due to Assumption 1, A_i is expressed in terms of the components of B_i:

$$A_i = \begin{bmatrix} \mu b_{ix} \\ \mu b_{iy} \\ 0 \end{bmatrix} \tag{22}$$

In order to find the required system of nonlinear algebraic equations, (23) is derived from (20):

$$\begin{aligned} L_i^2 = {} & A_i^T A_i + A_i^T R(p)^T P - A_i^T R(p)^T B_i + \\ & + P^T R(p) A_i + P^T P - P^T B_i - B_i^T R(p) A_i + \\ & - B_i^T P - B_i^T B_i \end{aligned} \tag{23}$$

Then, the vectors shown in (24) can be obtained from (23) in order to derive the system of nonlinear equations to find the unique rotation quaternion and its equivalent rotation matrix:

$$\begin{aligned} W_{i1} &= A_i^T + P^T R(p) - B_i^T R(p) \\ W_{i2} &= A_i^T R(p)^T + P^T - B_i^T \\ W_{i3} &= -A_i^T R(p)^T - P^T \\ W_{i4} &= -B_i^T \end{aligned} \tag{24}$$

for $i = 1, \ldots, 6$. Finally, with the formulas explained in (24), the system of nonlinear algebraic equation (25) can be obtained resolving for the components of the quaternion p, which represents the orientation of the platform:

$$L_i^2 = W_{i1} A_i + W_{i2} P + W_{i3} B_i + W_{i4} B_i \tag{25}$$

for $i = 1, \ldots, 6$. Equation (25) corresponds to a system of 6×4 (six equations with four variables) nonlinear algebraic equations. It is necessary to solve the system for the quaternion components p_0, p_1, p_2, p_3 in order to obtain the quaternion, which represents the orientation of the top platform. The vectors W_{i1}, W_{i2}, W_{i3}, W_{i4} in terms of the quaternion p are defined in the Appendix.

3. STEWART PLATFORM VELOCITIES BY UNIT QUATERNIONS

In this section, the angular and linear velocity of the platform along with the velocities of the legs are derived to extend the kinematic analysis of the Stewart–Gough platform, considering the results of the previous section.

3.1. Linear and angular velocity of platform

To derive the linear velocity of the platform, consider the vector located at the center of gravity of the upper platform in the upper frame (x', y', z'), represented by V_{cg} as shown in Figure 1. The components of V_{cg} are:

$$V_{cg} = \begin{bmatrix} V_{cgx} \\ V_{cgy} \\ V_{cgz} \end{bmatrix} \tag{26}$$

and L_{cg} is obtained by:

$$L_{cg} = R(p)V_{cg} + P \tag{27}$$

where $R(p)$ is the rotation matrix in terms of quaternions shown in (4) and P is the vector in (16). Therefore, substituting these equations in (27) and reorganizing, yields:

$$\begin{aligned} L_{cg} &= \left(p \otimes d_x \otimes p^*\right)V_{cgx} + \\ &+ \left(p \otimes d_y \otimes p^*\right)V_{cgy} + \\ &+ \left(p \otimes d_z \otimes p^*\right)\left(V_{cgz} + r\right) \end{aligned} \tag{28}$$

The following property is important to derive the velocity of the platform (Chou 1992; Spong *et al.* 2006):

Property 1: Consider unit quaternion p and p^*, which represents a rotation by the Euler angle. Then, vector quaternion d_r^0 for $r = x, y, z$ is obtained from d_r:

$$d_r^0 = p \otimes d_r \otimes p^* \tag{29}$$

Using Property 1, d_r can be obtained from d_r^0 in the form:

$$d_r = p^* \otimes d_r^0 \otimes p \tag{30}$$

The equivalences in (30) are used later to derive the linear velocity of the center of gravity of the platform. Now, taking the derivative of (28) the linear velocity of the top platform is obtained as shown in (31):

$$\begin{aligned} \dot{L}_{cg} &= \left(\dot{p} \otimes d_x \otimes p^* + p \otimes d_x \otimes \dot{p}^* + p \otimes \dot{d}_x \otimes p^*\right)V_{cgx} + \\ &+ \left(\dot{p} \otimes d_y \otimes p^* + p \otimes d_y \otimes \dot{p}^* + p \otimes \dot{d}_y \otimes p^*\right)V_{cgy} + \\ &+ \left(\dot{p} \otimes d_z \otimes p^* + p \otimes d_z \otimes \dot{p}^* + p \otimes \dot{d}_z \otimes p^*\right)\left(V_{cgz} + r\right) \end{aligned} \tag{31}$$

Susbtituting (30) in (31) for $r = x, y, z$, and considering that $\dot{d}_r = 0$, the following equation is obtained (Chou 1992):

$$\begin{aligned} \dot{L}_{cg} &= \left(\dot{p} \otimes p^* \otimes d_x^0 + d_x^0 \otimes p \otimes \dot{p}^*\right)V_{cgx} + \\ &+ \left(\dot{p} \otimes p^* \otimes d_y^0 + d_y^0 \otimes p \otimes \dot{p}^*\right)V_{cgy} + \\ &+ \left(\dot{p} \otimes p^* \otimes d_z^0 + d_z^0 \otimes p \otimes \dot{p}^*\right)\left(V_{cgz} + r\right) \end{aligned} \tag{32}$$

The following property is used to reduce (32) (Chou, 1992):

Property 2: Consider the unit quaternion p and its conjugate p^*, which represents a rotation by the Euler angle. From Property 1, the following equivalence is obtained:

$$\begin{aligned} \gamma &= \dot{p} \otimes p^* = -p \otimes \dot{p}^* \\ \rho &= \dot{p}^* \otimes p = -p^* \otimes \dot{p} \end{aligned} \tag{33}$$

Using Property 2 in (32) yields:

$$\begin{aligned} Vel_{cg} = \dot{L}_{cg} &= \left(\gamma \otimes d_x^0 - d_x^0 \otimes \gamma\right)V_{cgx} + \\ &+ \left(\gamma \otimes d_y^0 - d_y^0 \otimes \gamma\right)V_{cgy} + \\ &+ \left(\gamma \otimes d_z^0 - d_z^0 \otimes \gamma\right)\left(V_{cgz} + r\right) \end{aligned} \tag{34}$$

Due to the fact that γ is a vector quaternion (Funda *et al.* 1990; Chou 1992):

$$\gamma \otimes d_r^0 = -\vec{C} \cdot \vec{d}_r^{\,0} + \vec{C} \times \vec{d}_r^{\,0} \tag{35}$$

where $\vec{C}$ and $\vec{d}_r^0$ are the vector parts of the quaternions γ and d_r^0, respectively. The operations of the form:

$$m = \gamma \otimes d_r^0 - d_r^0 \otimes \gamma \tag{36}$$

can be reduced due to γ and d_r^0 being vector quaternions as shown in (35). Making the quaternion operations of (36) yields:

$$m = \left(-\vec{C}\cdot\vec{d}_r^0 + \vec{C}\times\vec{d}_r^0\right) - \left(-\vec{d}_r^0\cdot\vec{C} + \vec{d}_r^0\times\vec{C}\right) \tag{37}$$

Then, simplifying (37) and using the property of cross product $A \times B = -B \times A$ the equations (36) take the following form:

$$m = 2\vec{C}\times\vec{d}_r^0 \tag{38}$$

for $r = x, y, z$. Then, the linear velocity of platform (34) is finally given in (39), by using (38):

$$Vel_{cg} = \left(2\vec{C}\times\vec{d}_x^0\right)V_{cgx} + \left(2\vec{C}\times\vec{d}_y^0\right)V_{cgy} + \\ + \left(2\vec{C}\times\vec{d}_z^0\right)\left(V_{cgz} + r\right) \tag{39}$$

where vector quaternions γ and d_r^0 are:

$$\gamma = [0, c_1, c_2, c_3]^T \\ d_r^0 = \left[0, d_{r1}^0, d_{r2}^0, d_{r3}^0\right]^T \tag{40}$$

It is important to notice that (39) can also be obtained in matrix form. The reader can refer to (Chou 1992) to find a matrix form representation of this formula.

The angular velocity of the top platform can be yielded from (39), noticing that the linear velocity is $V = \omega \times r$ with radius r, or from (38) as a matrix form. Then, the angular velocity of the platform derived from (39) is finally given by:

$$\omega_{cg} = 2\vec{C} \tag{41}$$

or in vector quaternion form:

$$\omega_{cg} = 2\gamma \tag{42}$$

3.2. Velocity of legs

As noticed, the derivations of the linear leg velocities by unit quaternions are very straightforward and follow a similar procedure to the platform velocity. In this subsection, vector V_{bi} represents the leg vectors and A_i, for $i = 1, \ldots, 6$, has previously been defined to obtain the leg linear velocity, and it is given by (see Fig. 1) (Ji, Wu 2001; Wang *et al.* 2011):

$$V_{bi} = R(p)A_i + P - B_i \tag{43}$$

Then, deriving (43) and considering that the derivative of B_i is a zero vector, the following formula is obtained:

$$\dot{V}_{bi} = \dot{R}(p)A_i + \dot{P} \tag{44}$$

Now, consider that the following condition:

$$R(p)R(p)^T = I \tag{45}$$

holds for the rotation matrix with quaternions components, where I is the identity matrix (Spong *et al.* 2006). Deriving (45) yields:

$$\dot{R}(p)R(p)^T + R(p)\dot{R}(p)^T = 0 \tag{46}$$

Define $S\left[\omega(t)\right]$ as (Spong *et al.* 2006):

$$S\left[\omega(t)\right] = \left(\dot{R}(p)\right)R(p)^T \tag{47}$$

The transpose of (47) is:

$$S^T\left[\omega(t)\right] = R(p)\left(\dot{R}(p)^T\right) \tag{48}$$

where $S\left[\omega(t)\right]$ is a skew symmetric matrix. Then, using this matrix, the following result can be obtained for linear leg velocity (44) using unit quaternions:

$$\dot{V}_{bi} = S\left[\omega(t)\right]R(p)A_i + \dot{P} \tag{49}$$

From equation (49), the linear velocities of the legs are obtained in each frame.

4. NUMERICAL EXAMPLES

In this section, a series of numerical examples are shown to test the theoretical background explained in this paper. The two examples, in which unit quaternions are implemented to derive the Stewart platform kinematics, are:

1) Forward kinematic analysis of the Stewart platform,
2) Linear and angular velocity of the Stewart platform.

The parameters of Stewart platform used in these examples are shown in Table 1.

Table 1

Parameters of Stewart platform

r_l [m]	0.3
r_u [m]	0.2
r [m]	0.1

The components of the inertia tensor $I_{xy}, I_{xz}, I_{yz}, I_{yx}, I_{zx}, I_{zy}$, are zero. These values and the mass of the top platform are found using *SOLIDWORKS*, and all the simulations have been done with *MATLAB*.

4.1. Example 1

In this subsection the forward kinematics using unit quaternions as explained in Section 2 is performed using three series of leg lengths in order to find the orientation of the platform with its respective rotation matrix. The three series of leg lengths are shown in Table 2. Then, the results are compared with a rotation matrix given by Euler angles as shown in (Lopes 2009).

Table 2

Leg length series

Leg length	Series 1 [m]	Series 2 [m]	Series 3 [m]
L_1	0.486	0.592	0.876
L_2	0.518	0.621	0.985
L_3	0.484	0.595	0.897
L_4	0.513	0.624	1.010
L_5	0.477	0.596	0.911
L_6	0.511	0.624	1.006

The unit quaternions and rotation matrices, that represent the orientation of the Stewart platform for the three series, are obtained by solving numerically a set of nonlinear algebraic equations (25), and the same is done for the rotation matrix (Lopes 2009). It is used for comparison purposes and is given by $R = R_z(\gamma)R_y(\beta)R_x(\alpha)$ where α, β, γ are the respective Euler angles. Consider that the performance index obtained by solving numerically the proposed approach is the nonlinear algebraic equation solver evaluation function. It is proved later that due to the simplicity of quaternions the results are more accurate in comparison with the results shown in (Lopes 2009).

The rotation matrix and unit quaternion that represent the orientation of the Stewart platform for series 1 are given respectively by:

$$R(p)=\begin{bmatrix}1.780492 & -0.266144 & -0.026571\\ 0.265719 & 1.780522 & -0.028751\\ 0.030527 & 0.024511 & 1.800044\end{bmatrix} \quad (50)$$

$$p=[1.3380515\ \ 0.0099515\ \ -0.0106680\ \ 0.0993727]^T \quad (51)$$

Then, the orientation of the Stewart platform, given the unit quaternion (51) and rotation matrix (50), obtained by the proposed approach, is depicted in Figure 2.

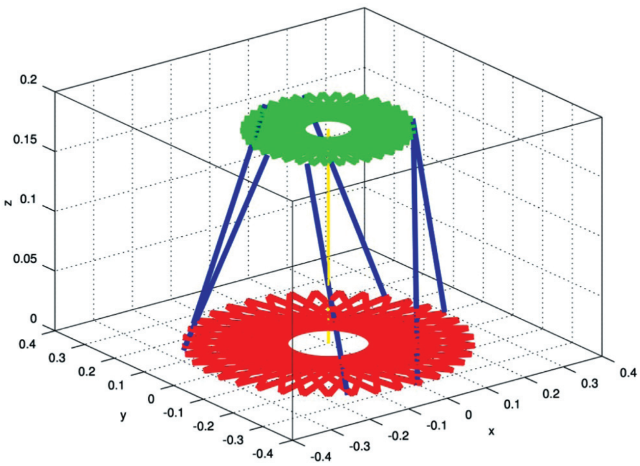

Fig. 2. Orientation of the platform for series 1 (proposed approach)

It can be noticed that there is a little difference in the orientation of the platform due to some accuracy issues in the solver, needed to find the solution of the nonlinear algebraic equations for the given leg lengths of series 1. In Figure 3, the orientation of the platform for series 1, obtained with the compared approach, is very similar as shown in Figure 2 but, as can be noticed in Table 3, the results are more accurate in comparison with the compared approach.

The rotation matrix and unit quaternion found for series 2, are given, respectively, below:

$$R(p)=\begin{bmatrix}-0.679737 & -0.432079 & 0.030406\\ 0.432709 & -0.679918 & 0.011523\\ 0.019472 & 0.026041 & 0.805358\end{bmatrix} \quad (52)$$

$$p=[0.250857\quad 0.014469\quad 0.010897\quad 0.861833]^T \quad (53)$$

The orientation of the Stewart platform for series 2 obtained by the proposed approach, is shown in Figure 4.

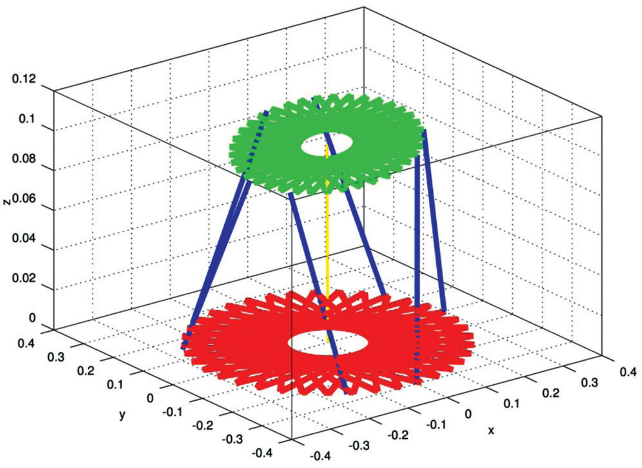

Fig. 3. Orientation of the platform for series 1 (compared approach)

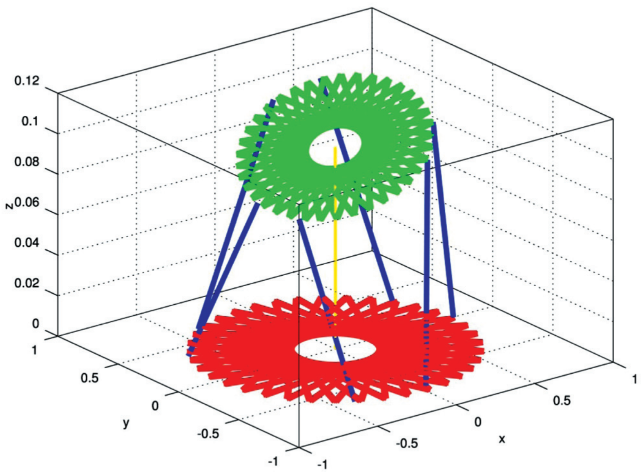

Fig. 5. Orientation of the platform for series 2 (compared approach)

Table 3

Nonlinear algebraic equation solver error

Series	Solver error (proposed approach)	Solver error (compared approach)
1	0.0075	0.8873
2	0.0018	0.8416
3	0.1854	0.0910

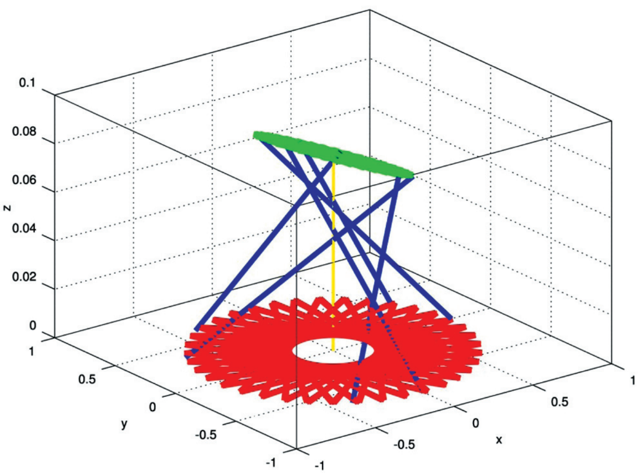

Fig. 4. Orientation of the platform for series 2 (proposed approach)

The orientation of the platform for series 2 has some differences due to the accuracy of the solver of nonlinear algebraic equations (25). In Figure 5 the orientation of the platform for series 2 is shown and, as can be noticed, there are some differences between Figure 4 and Figure 5. The compared approach is less accurate and more complex than the proposed approach, as shown in Table 3.

The rotation matrix and unit quaternion which represent the orientation of the Stewart platform for series 3 are given by:

$$R(p)=\begin{bmatrix} 2.53046 & -6.01070 & 0.90749 \\ 6.03694 & 2.60006 & 0.38782 \\ -0.71237 & 0.68299 & 6.51010 \end{bmatrix} \quad (54)$$

$$p=[2.134543 \quad 0.034571 \quad 0.189721 \quad 1.41103] \quad (55)$$

The orientation of the Stewart platform for series 3 for the proposed approach is depicted in Figure 6.

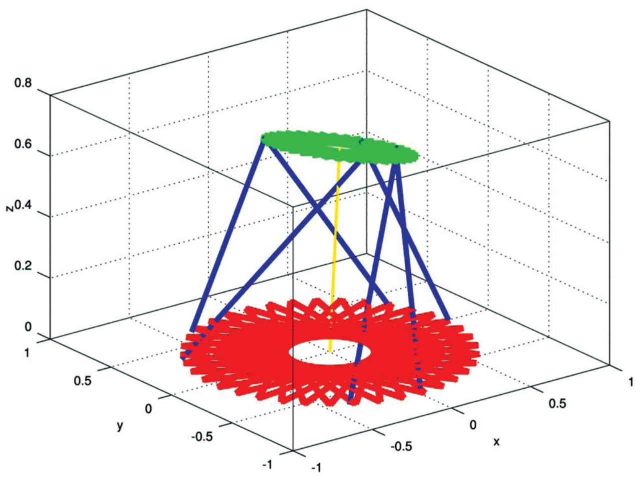

Fig. 6. Orientation of the platform for series 3 (proposed approach)

As it is noticed in Figure 6, even when the orientation of the platform is accurate, there are some differences, but the solution of nonlinear algebraic equations is quite equivalent to the true solution. Finally, the rotation matrix multiplication $R^T(p)R(p)$ is not equal to the identity matrix, but it is a positive diagonal matrix approximate to the identity matrix. In comparison with Figure 7, the results shown in Figure 6 depict a more approximate orientation even when the solver error is greater than the compared approach. However, due to the simplicity of quaternions, the complexity of the systems of nonlinear algebraic equations is reduced and more accurate results are obtained. As a conclusion, the forward kinematics of the Stewart platform can be obtained easily by unit quaternions solving a unique solution of the rotation matrix in comparison with multiple solutions of the forward kinematic problem, as other methods found in the literature.

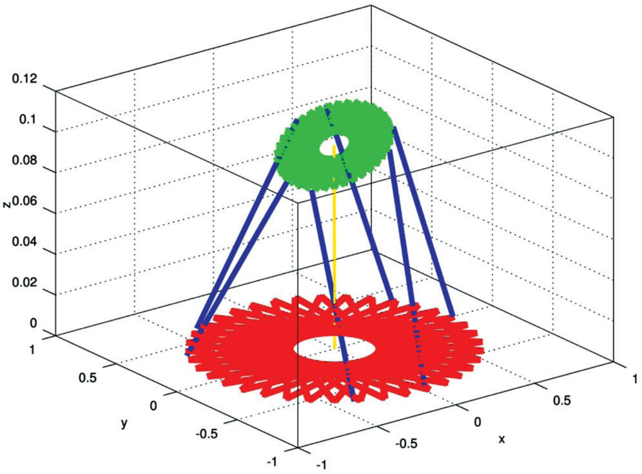

Fig. 7. Orientation of the platform for series 3 (compared approach)

4.2. Example 2

In this section, the linear and angular velocities of the center of gravity of the platform are analyzed in the range of the Euler angle $0 \leq \theta \leq 2\pi$. For this purpose, the center of gravity of the top frame (x', y', z'), in this example, is given by:

$$V_{cg} = [0 \quad 0.0025 \quad 0] \tag{56}$$

and the vector part of the unit quaternion p:

$$\vec{p}^T = [2.5 \quad 10 \quad 14] \tag{57}$$

Then, the linear velocity Vel_{cg} of the center of gravity (top platform) is shown in Figure 8 for the Euler angle in the range $0 \leq \theta \leq 2\pi$.

In Figure 8 a parametric plot of the linear velocity of the platform Vel_{cg} is shown while varying the Euler angle $0 \leq \theta \leq 2\pi$.

In Figure 9, a parametric plot of the angular velocity ω_{cg} of the Stewart platform is shown. The components of the angular velocity ω_{cgx}, ω_{cgy} and ω_{cgz} are depicted in this figure. It can be noticed that the angular velocity of the center of mass of the Stewart platform follows a similar trajectory as the linear velocity of the platform. The plots shown in Figure 8 and Figure 9 give an idea on how the linear and angular velocity of the gravity center of the platform behaves when the Euler angle θ varies. The implementation of Euler angle rotations, using unit quaternions, has several advantages over other methods to describe the linear and angular velocity of the platform, because of the simplicity of unit quaternions and the possibility to obtain an equivalent rotation matrix to specify the orientation of the platform.

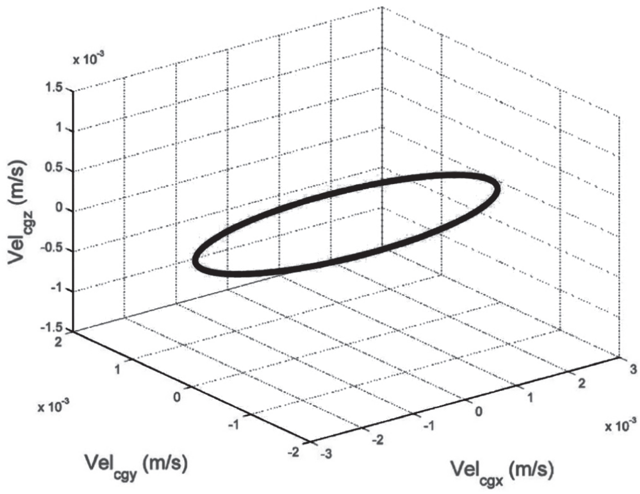

Fig. 8. Linear velocity of the platform while varying $0 \leq \theta \leq 2\pi$

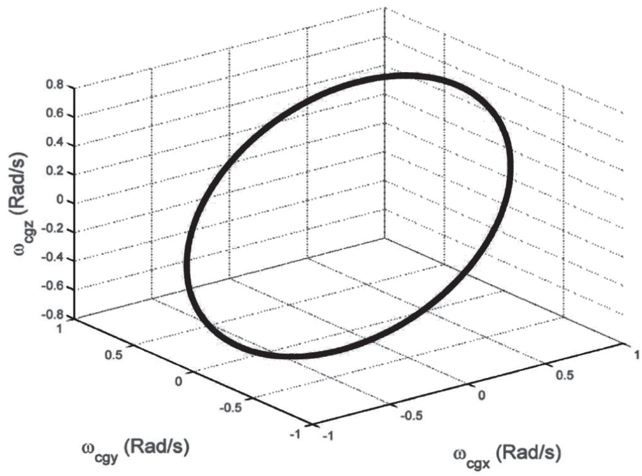

Fig. 9. Angular velocity of the platform while varying $0 \leq \theta \leq 2\pi$

The results obtained in this subsection show the relation of the three components of the linear and angular velocity of the platform. The plots in Figures 8 and 9 provide valuable information regarding these variables. The use of the equations for Vel_{cg} and ω_{cg} are useful to compute the linear and angular velocities for the three axes.

5. CONCLUSIONS

In this paper a complete kinematic analysis of the Stewart–Gough platform using unit quaternions is shown, considering the advantages of this mathematical representation given by Euler angles. Unlike other works found in the literature, not only the forward kinematics of the Stewart platform is derived in this paper, but other variables are also presented (such as platform velocity and legs velocities). It has been proved, theoretically and numerically, that a suitable forward and inverse kinematics model of the Stewart platform can be derived by implementing unit quaternions and then, for the forward kinematics case, the orientation of the platform can be found by solving a system of nonlinear algebraic equations numerically to obtain the orientation of the platform. The velocities of the platform are deduced by a suitable mathematical model derived by unit quaternions. A simpler and less computationally complex model is used to obtain these variables. The leg velocities are derived using the rotation matrix given by unit quaternions. Because of the simplicity of quaternions to represent rotations, these variables can be found efficiently by an accurate mathematical model. As a future direction of this work, this study will be extended to the derivations of a dynamic model starting from the Stewart–Gough kinematics in order to derive efficient control strategies for this mechanism implemented in applied research projects.

Appendix

In this section, the components of the vectors W_{i1}, W_{i2}, W_{i3} and W_{i4} are detailed. Each vector W_{ir} has components $W_{ir}(j)$ for $r = 1, \ldots, 4$ and $j = 1, \ldots, 3$ given below.

$$
\begin{aligned}
W_{i1}(1) &= \mu b_{ix} + rp_x\left(p_1^2 + p_0^2 - p_3^2 - p_2^2\right) + \\
&\quad + rp_y(2p_1p_2 + 2p_0p_3) + \\
&\quad + rp_z(-2p_0p_2 + 2p_1p_3) \\
W_{i1}(2) &= \mu b_{iy} + rp_x(-2p_0p_3 + 2p_1p_2) + \\
&\quad + rp_y\left(p_2^2 + p_0^2 - p_1^2 - p_3^2\right) + \\
&\quad + rp_z(2p_2p_3 + 2p_1p_0) \\
W_{i1}(3) &= rp_x(2p_1p_3 + 2p_0p_2) + \\
&\quad + rp_y(-2p_1p_0 + 2p_2p_3) + \\
&\quad + rp_z\left(p_3^2 + p_0^2 - p_2^2 - p_1^2\right)
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
W_{i2}(1) &= \mu b_{ix}\left(p_1^2 + p_0^2 - p_3^2 - p_2^2\right) + \\
&\quad + \mu b_{iy}(-2p_0p_3 + 2p_1p_2) + rp_x \\
W_{i2}(2) &= \mu b_{ix}(2p_1p_2 + 2p_0p_3) + \\
&\quad + \mu b_{iy}\left(p_2^2 + p_0^2 - p_1^2 - p_3^2\right) + rp_y \\
W_{i2}(3) &= \mu b_{ix}(-2p_0p_2 + 2p_1p_3) + \\
&\quad + \mu b_{iy}(2p_2p_3 + 2p_1p_0) + rp_z
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
W_{i3}(1) &= -\mu b_{ix}\left(p_1^2 + p_0^2 - p_3^2 - p_2^2\right) - \\
&\quad + \mu b_{iy}(-2p_0p_3 + 2p_1p_2) - rp_x \\
W_{i3}(2) &= -\mu b_{ix}(2p_1p_2 + 2p_0p_3) - \\
&\quad + \mu b_{iy}\left(p_2^2 + p_0^2 - p_1^2 - p_3^2\right) - rp_y \\
W_{i3}(3) &= -\mu b_{ix}(-2p_0p_2 + 2p_1p_3) - \\
&\quad + \mu b_{iy}(2p_2p_3 + 2p_1p_0) - rp_z
\end{aligned}
\tag{60}
$$

and finally:

$$
W_{i4} = \begin{bmatrix} -b_{ix} & -b_{iy} & 0 \end{bmatrix}
\tag{61}
$$

where:

$$
\begin{aligned}
p_x &= 2p_1p_3 + 2p_0p_2 p_y = \\
&= -2p_1p_0 + 2p_2p_3 p_z = p_3^2 + p_0^2 - p_2^2 - p_1^2
\end{aligned}
\tag{62}
$$

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References

Campa R., Camarillo K., Arias L., 2006, *Kinematic modeling and control of robot manipulators via unit quaternions: Application to a spherical wrist.* 45th IEEE Conference on Decision and Control 2006, 6474–6479.

Cao Y., Zhou H., Ji W., Liu M., Liu X., 2010, *Orientation--singularity and orientation capability analysis of Stewart platform based on unit quaternion representation.* 2010 IEEE International Conference on Mechatronics and Automation, 452–457.

Chen C.-T., Renn J.-C., Yan Z.-Y., 2011, *Experimental identification of inertial and friction parameters for electro-hydraulic motion simulators*. Mechatronics 21(1), 1–10.

Chen S.-H., Fu, L.-C., 2006, *The forward kinematics of the 6-6 Stewart platform using extra sensors*. 2006 IEEE International Conference on Systems, Man and Cybernetics, 6, 4671–4676.

Chen S.-H., Fu L.-C., 2008, *Output feedback control with a nonlinear observer based forward kinematics solution of a Stewart platform*. 2008 IEEE International Conference on Systems, Man and Cybernetics, 3150–3155.

Chen S.-H., Fu L.-C., 2013, *Output feedback sliding mode control for a Stewart platform with a nonlinear observer-based forward kinematics solution*. IEEE Transactions on Control Systems Technology, 21(1), 176–185.

Choi M., Kim W., Yi B.-J., 2007, *Trajectory planning in 6-degrees--of-freedom operational space for the 3-degrees-of-freedom mechanism conqured by constraining the Stewart platform structure*. 2007 International Conference on Control, Automation and Systems, 1222–1227.

Chou J., 1992, *Quaternion kinematic and dynamic differential equations*. IEEE Transactions on Robotics and Automation 8(1), 53–64.

Dongya Z., Shaoyuan L., Feng G., 2007, *Continuous finite time control for Stewart platform with terminal sliding mode*. 2007 Chinese Control Conference, 27–30.

Duindam V., Stramigioli S., 2008, *Singularity-free dynamic equations of open-chain mechanisms with general holonomic and nonholonomic joints*. IEEE Transactions on Robotics 24(3), 517–526.

Fresk E., Nikolakopoulos G., 2013, *Full quaternion based attitude control for a quadrotor*. 2013 European Control Conference (ECC), 3864–3869.

Funda J., Paul R., 1990, *A computational analysis of screw transformations in robotics*. IEEE Transactions on Robotics and Automation 6(3), 348–356.

Funda J., Taylor R., Paul R., 1990, *On homogeneous transforms, quaternions, and computational efficiency*. IEEE Transactions on Robotics and Automation 6(3), 382–388.

Ghobakhloo A., Eghtesad M., Azadi M., 2006, *Adaptive-robust control of the Stewart–Gough platform as a six DOF parallel robot*. 2006 World Automation Congress, 1–6.

He R., Zhao Y., Yang S., Yang S., 2010, *Kinematic-parameter identification for serial-robot calibration based on POE formula*. IEEE Transactions on Robotics 26(3), 411–423.

Huang C.-I., Fu L.-C., 2004, *Adaptive backstepping tracking control of the Stewart platform*. 2004 43rd IEEE Conference on Decision and Control (CDC), 5, 5228–5233.

Huang C.-I., Fu L.-C., 2005, *Smooth sliding mode tracking control of the Stewart platform*. Proceedings of 2005 IEEE Conference on Control Applications (CCA 2005), 43–48.

Ji P., Wu H., 2001, *A closed-form forward kinematics solution for the 6-6P Stewart platform*. IEEE Transactions on Robotics and Automation 17(4), 522–526.

Liu M.-J., Li C.-X., Li C.-N., 2000, *Dynamics analysis of the Gough–Stewart platform manipulator*. IEEE Transactions on Robotics and Automation 16(1), 94–98.

Lopes A.M., 2009, *Dynamic modeling of a Stewart Platform using the generalized momentum approach*. Communication in Nonlinear Science Numerical Simulation 14, 3389–3401.

Lou Y., Zhang Y., Huang R., Chen X., Li Z., 2014, *Optimization algorithms for kinematically optimal design of parallel manipulators*. IEEE Transactions on Automation Science and Engineering 11(2), 574–584.

Morell A., Acosta L., Toledo J., 2012, *An artificial intelligence approach to forward kinematics of Stewart platforms*. 2012 20th Mediterranean Conference on Control and Automation (MED), 433–438.

Morell A., Tarokh M., Acosta L., 2013, *Solving the forward kinematics problem in parallel robots using support vector regression*. Engineering Applications of Artificial Intelligence 26(7), 1698–1706.

Nanua P., Waldron K., 1989, *Direct kinematic solution of a Stewart platform*. Proceedings of the 1989 IEEE International Conference on Robotics and Automation, 431–437.

Omran A., Kassem A., 2011, *Optimal task space control design of a Stewart manipulator for aircraft stall recovery*. Aerospace Science and Technology 15(5), 353–365.

Portman V., Chapsky V., Shneor Y., 2012, *Workspace of parallel kinematics machines with minimum stiffness limits: Collinear stiffness value based approach.* Mechanism and Machine Theory 49, 67–86.

Quoc L.H., Thanh N.M., 2013, *Definition of linearly dependent screws in singularity configurations of parallel mechanisms and experimental based on computing of the system*. 2013 13th International Conference on Control, Automation and Systems (ICCAS), 1100–1107.

Spong M.W., Hutchinson S., Vidyasagar M., 2006, *Robot Modeling and Control*. John Wiley and Sons.

Su Y., Zheng C., Duan B.Y., 2002, *Singularity analysis of a 6 DOF Stewart platform using genetic algorithm.* IEEE International Conference on Systems, Man and Cybernetics 7.

Tari H., Su H.-J., Hauenstein J., 2012, *Classification and complete solution of the kinetostatics of a compliant Stewart–Gough platform*. Mechanism and Machine Theory 49, 177–186.

Tu K.-Y., Wu T.-C., Lee T.-T., 2004, *A study of Stewart platform specifications for motion cueing systems*. 2004 IEEE International Conference on Systems, Man and Cybernetics 7, 3950–3955.

Wang Z., He J., Gu H., 2011, *Forward kinematics analysis of a six--degree-of-freedom Stewart platform based on independent component analysis and Nelder–Mead algorithm*. IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans 41(3), 589–597.

Yang C., Zheng S., Lan X., Han J., 2011, *Adaptive robust control for spatial hydraulic parallel industrial robot*. Procedia Engineering 15, 331–335.

Zhang G.-F., 2012, *Classification of direct kinematics to planar generalized Stewart platforms*. Computational Geometry 45(8), 458–473.