

COMPUTATIONAL MODELLING OF LOCALIZED DEFORMATIONS WITH REGULARIZED CONTINUUM MODELS

SUMMARY

The paper presents a short overview of selected problems related to the numerical analysis of localized deformations. After defining the localization phenomenon and the class of gradient models, two simulation examples are shown. They are applications of the plasticity theory with a Laplacian of the hardening parameter and of the damage theory with an additional averaging equation for an equivalent strain measure.

Keywords: strain localization, gradient plasticity, gradient damage, finite element method

MODELOWANIE KOMPUTEROWE DEFORMACJI ZLOKALIZOWANYCH ZREGULARYZOWANYMI MODELAMI OŚRODKA CIĄGŁEGO

Artykuł przedstawia krótki przegląd wybranych problemów analizy numerycznej deformacji zlokalizowanych. Po zdefiniowaniu zjawiska lokalizacji i klasy modeli gradientowych zwięźle opisano dwa przykładowe zastosowania teorii płynięcia plastycznego zawierającej laplasjan parametru wzmocnienia i teorii uszkodzenia z dodatkowym równaniem uśredniającym miarę odkształcenia w otoczeniu punktu.

Słowa kluczowe: lokalizacja odkształceń, gradientowa plastyczność, gradientowa mechanika uszkodzenia, metoda elementów skończonych

1. INTRODUCTION

Strain localization is one of fundamental problems in computer simulation of mechanical behaviour of materials with microstructure, in particular those important in civil engineering (geomaterials, including concrete as “artificial rock”) and mechanical engineering (various composites). As is well-known, localized deformation is closely related to material softening which can lead to the loss of well-posedness of the (initial) boundary value problem if a classical continuum model is used. Although the phenomena of softening and localization are observed in macroscopic material response, they have their physical origin in the microstructure evolution.

In this overview article the following issues are concisely presented: definition of the problem of localized deformation, theoretical description of material instabilities and their consequences as well as discontinuity types occurring in the description of materials undergoing plastic softening, damage or fracture. Attention is focused on the problem of pathological sensitivity of numerical solution to the choice of discretization and on alternative regularization methods (de Borst *et al.* 1993; Sluys 1992). The methods of discontinuity representation in the finite element method (FEM) will be mentioned.

Two-dimensional examples of simulation of static localization are presented. In the analysis small strains and isothermal conditions are assumed. Moreover, rheologic effects are neglected. Gradient-enhanced isotropic models of plastic flow and continuum damage are used. They incorporate a nonlocal averaging procedure based on higher-order gra-

dients of an internal parameter and averaged strain measure, respectively. The theories contain intrinsic length parameters and the gradient terms represent nonlocal microstructure interactions in the phenomenological (macroscopic) modelling.

In particular, the gradient plasticity with Huber-Mises yield function is used to simulate a curved shear band formed in a slope stability problem, cf. (Pamin and de Borst 1995). Then, the gradient-enhanced damage model (Peerlings *et al.* 1996) is applied in the simulation of cracking in a reinforced concrete beamsubjected to four-point bending. It is mentioned that both the constitutive models were employed in (Pamin 2005) to simulate strain localization during wave propagation in a tensile bar. For an overview of dynamic strain localization phenomena, the Reader is referred to (Sluys 1992; Pedersen 2009). The present article is completed by some final remarks on the relevance of regularized continuum models.

2. STRAIN LOCALIZATION PHENOMENON

Many materials lose the ability to behave in a stable manner at some level of deformation. The consequence of this fact is strain localization which often leads to failure of a specimen or structure. Localization means that, in the loading history of a body, from a certain moment on the whole deformation is concentrated in a narrow band while the other parts of the body exhibit unloading. Typical examples of localized deformation of a continuous material configuration are a decohesion band and a shear band. Their simulated representations are shown in Figure 1 for a typical two-dimensional (plane

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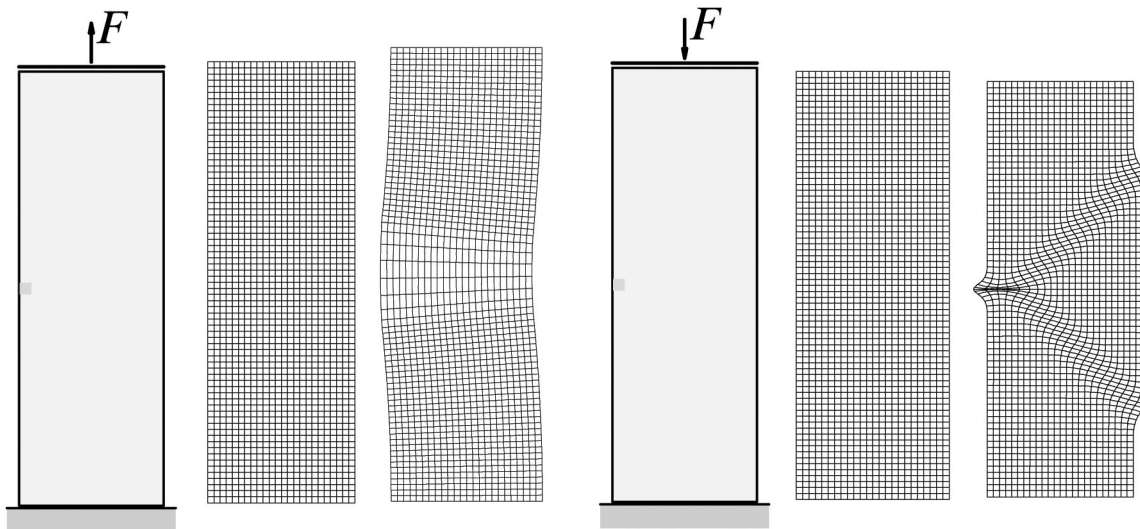


Fig. 1. Demonstration of strain localization in continuum modelling

strain) benchmark of a rectangular specimen. If tensile or compressive loading is applied at the upper edge under displacement control, the specimen is at first uniformly strained, but then a qualitative change in the response occurs and a localized deformation pattern is observed.

Materials which exhibit softening (e.g. concrete or soil) are often described in numerical analyses by a descending relation between stress and strain. In the classical formulation of continuum mechanics this can lead to ill-posed (initial) boundary value problem, since the equilibrium (or motion) equations do not remain elliptic (or respectively hyperbolic) once the descending branch is entered. Localization is then possible only in a so-called set of measure zero. In a two-dimensional case it is a line, which is illustrated in Figure 2. This means that localization is related to strain discontinuity across the line (called weak), which however can lead to displacement discontinuity (called strong).

As a consequence, in computations using numerical methods (e.g. FEM) one obtains results which are pathologi-

cally dependent on discretization, since the methods attempt to simulate localization in the smallest material volume admitted, and this volume depends on the adopted (finite element) mesh. The whole problem is concisely presented in Figure 3. It is emphasized that the same problem is present in all discretization methods, for instance in the (meshless) finite difference method the spacing of nodes determines the width of a simulated localization zone.

There are two manners in which the problem can be solved. One is to concentrate the deformation in interfaces or discrete cracks, i.e. strong discontinuity lines or surfaces, for which constitutive relations are written as a dependence of tractions on relative displacements of two interface sides. This can be done using so-called interface finite elements, see e.g. (Rots 1988). The approach is physically motivated for instance in the case of quasi-brittle materials which undergo macroscopic cracking. The use of interface elements requires one to predict the crack location or to assume the potential occurrence of cracks along all interelement

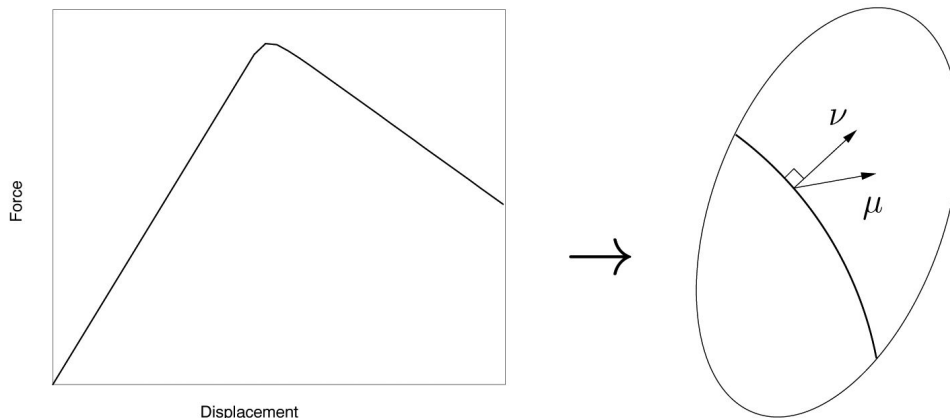


Fig. 2. Material instability as a necessary condition of ellipticity loss. Strain jump along discontinuity line

$$\llbracket \epsilon_{i,j} \rrbracket = \frac{1}{2} (v_i \mu_j + v_j \mu_i)$$

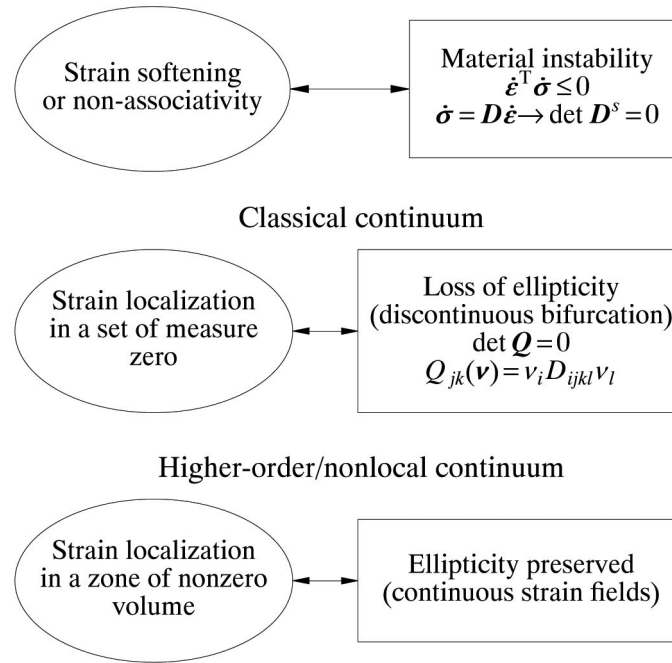


Fig. 3. Consequence of material instability and regularization ($\boldsymbol{\sigma}$ – stress tensor, $\boldsymbol{\varepsilon}$ – strain tensor, \mathbf{D} – material tangent operator, \mathbf{Q} – so-called acoustic tensor, \mathbf{v} – vector normal to discontinuity line)

boundaries of standard, continuum finite elements. Therefore, a more efficient concept was devised in the form of partition of unity elements called also extended finite elements (XFEM), see e.g. (Babuška and Melenk 1997; Wells 2001; Fries and Belytschko 2010). This concept enables one to simulate the formation and propagation of strong discontinuities through element domains owing to an enrichment of the approximation basis with Heaviside-type functions.

Another solution to the problem of ill-posedness of the mathematical model is to exploit enhanced (regularized) continuum theories. They have a nonlocal character and incorporate higher order deformation gradients or time rates in the constitutive description. These theories contain a so-called internal length scale which is an additional material parameter and defines a (non-zero) width of the localization band. The use of such enhanced continuum models removes the spurious mesh sensitivity of numerical simulation results. Hence, they enable a proper analysis of strain localization and material instabilities, but also of deterministic size effects observed in macroscopic problems (although they have their source in meso- or microstructure of the material).

The regularized models include the plastic flow theories which incorporate higher deformation gradients, see e.g. (Mühlhaus and Aifantis 1991; de Borst and Mühlhaus 1992; Vardoulakis and Sulem 1995; Svedberg and Runesson 1997; Liebe and Steinmann 2001; Engelen *et al.* 2003), and gradient theories of continuum damage, e.g. (Peerlings *et al.* 1996; Geers 1997; Comi 1999; Kuhl *et al.* 2000; Liebe *et al.* 2001; Askes *et al.* 2002). An alternative solution is provided by a micropolar continuum, e.g. (Mühlhaus and Vardoulakis

1987), rate dependent models, in particular viscoplastic, see for instance (Łodygowski 1998), and nonlocal integral models, e.g. (Rolshoven 2003). A short summary of the general concepts of nonlocal and gradient-enhanced continuum is given in the next section.

The advantages of the gradient approach are its versatility (micropolar regularization may be inactive in decohesion problems, viscoplastic in static ones) and the relative convenience with which they can be implemented in numerical methods (the nonlocal integral model requires an additional loop over elements in the FEM algorithm and a special treatment of boundaries). However, the usefulness of the regularized models is conditioned by the ability to determine the internal length parameter(s) from experiments or from a mechanical description of the heterogeneous material at a lower level of observation.

3. HIGHER ORDER CONTINUUM

The classical continuum (called “simple material”) has the property that stress $\boldsymbol{\sigma}$ at a point with Lagrange coordinate \mathbf{X} is at each time instant t a function of deformation gradient \mathbf{F} and possibly its history:

$$\boldsymbol{\sigma}(\mathbf{X}, t) = \boldsymbol{\psi}(\mathbf{X}, \mathbf{F}(\mathbf{X}, t)).$$

The deformation gradient has a standard definition:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t),$$

where \mathbf{x} is the motion function.

In a nonlocal continuum the stress $\boldsymbol{\sigma}$ at a point \mathbf{X} is a function of the motion history of points \mathbf{Z} in a certain vicinity of point \mathbf{X} or even in the whole body:

$$\boldsymbol{\sigma}(\mathbf{X}, t) = \boldsymbol{\psi}(\mathbf{X}, \mathbf{x}(\mathbf{Z}, t)).$$

Moreover, in so-called grade- n continuum (Becker and Bürger 1975) the stress $\boldsymbol{\sigma}$ at a point \mathbf{X} depends on higher than the first deformation gradients up to order n :

$$\boldsymbol{\sigma}(\mathbf{X}, t) = \boldsymbol{\psi}(\mathbf{X}, \mathbf{F}(\mathbf{X}, t), \mathbf{F}_2(\mathbf{X}, t), \dots),$$

where $\mathbf{F}_2 = \frac{\partial \mathbf{F}}{\partial \mathbf{X}}$. The gradient continuum model can be viewed as an approximation of a general nonlocal continuum where integral averaging is performed.

In the description of the gradient-enhanced (as well as nonlocal) continuum an internal length parameter l occurs, which determines the range of nonlocal interactions of microstructural deformation carriers. A classical continuum model can be applied as long as $l/L \ll 1$, where L describes a structural size or a characteristic wave length of deformation field. A higher-order continuum description is relevant when $l/L \rightarrow 1$. This is the case for localized deformations or various forms of microstructure evolution related to size effects, see e.g. (Fleck and Hutchinson 1997).

Different gradient-enhanced models have been formulated in the past, starting from the works of Mindlin, e.g. (Mindlin 1965). Some of them have been referred to in Sect. 2 and a broader review is provided in (Pamin 2004). With a growing interest in physically nonlinear mechanics, gradient versions of plastic flow and continuum damage theories became important. Such models are used in the examples described in the next section.

4. EXAMPLE SIMULATIONS OF LOCALIZED DEFORMATIONS

The gradient plasticity theory, originated in papers of Aifantis, de Borst and Mühlhaus, e.g. (Mühlhaus and Aifantis 1991; de Borst and Mühlhaus 1992), contains a yield function dependent on the Laplacian of a hardening parameter. It is applicable for materials which exhibit softening and/or non-associated plasticity. The gradient-dependence causes the plastic consistency condition to become a differential equation and makes it necessary to discretize the plastic multiplier next to the standard discretization of displacements. Hence, two-field finite elements are formulated.

An example of gradient plasticity application is shown in Figures 4 and 5. This is a problem of slope stability under increasing vertical displacement imposed by a stiff stamp. The gradient-enhanced Huber-Mises plasticity is used for simplicity, but in (Pamin and de Borst 1995; Stankiewicz 2007) the gradient versions of Drucker-Prager and Cam-clay plastic flow theories, respectively, were applied for a similar purpose.

The height of the analyzed configuration is 10 m, the slope inclination is $7/20$. The following material model data are assumed: linear elasticity $E = 250\,000 \text{ kN/m}^2$, $\nu = 0.25$, yield strength $\sigma_y = 4.0 \text{ kN/m}^2$, hardening modulus $h = -16000 \text{ kN/m}^2$, internal length scale $l = 0.4 \text{ m}$. It is assumed that the stamp is made from a material having 100 times larger Young's modulus and no sliding between the stamp and the slope is admitted. In computations three meshes are used, 10×10 , 20×20 i 40×40 eight-noded gradient-plasticity finite elements with 2×2 Gaussian integration (de Borst and Pamin 1996). The kinematic boundary conditions are shown in Figure 4, the loading is applied under the vertical displacement control to the upper edge of the stamp. Homogeneous natural boundary conditions are adopted for the additionally discretized plastic multiplier field.

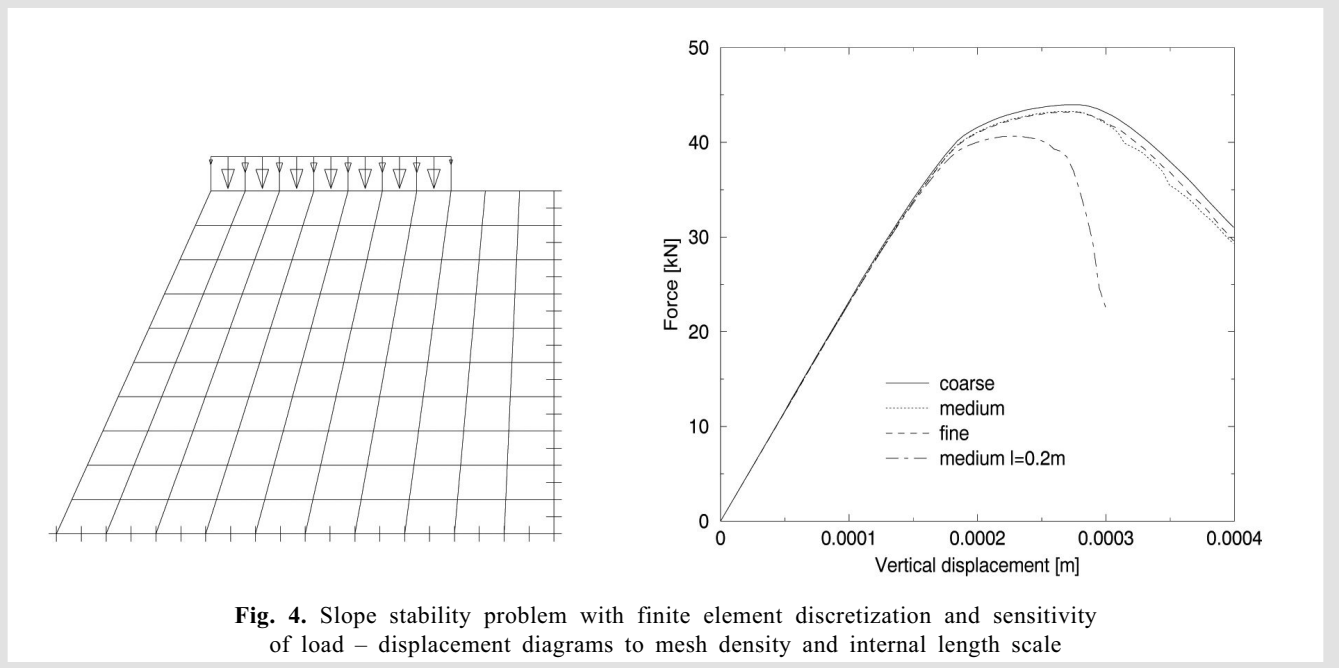


Fig. 4. Slope stability problem with finite element discretization and sensitivity of load – displacement diagrams to mesh density and internal length scale

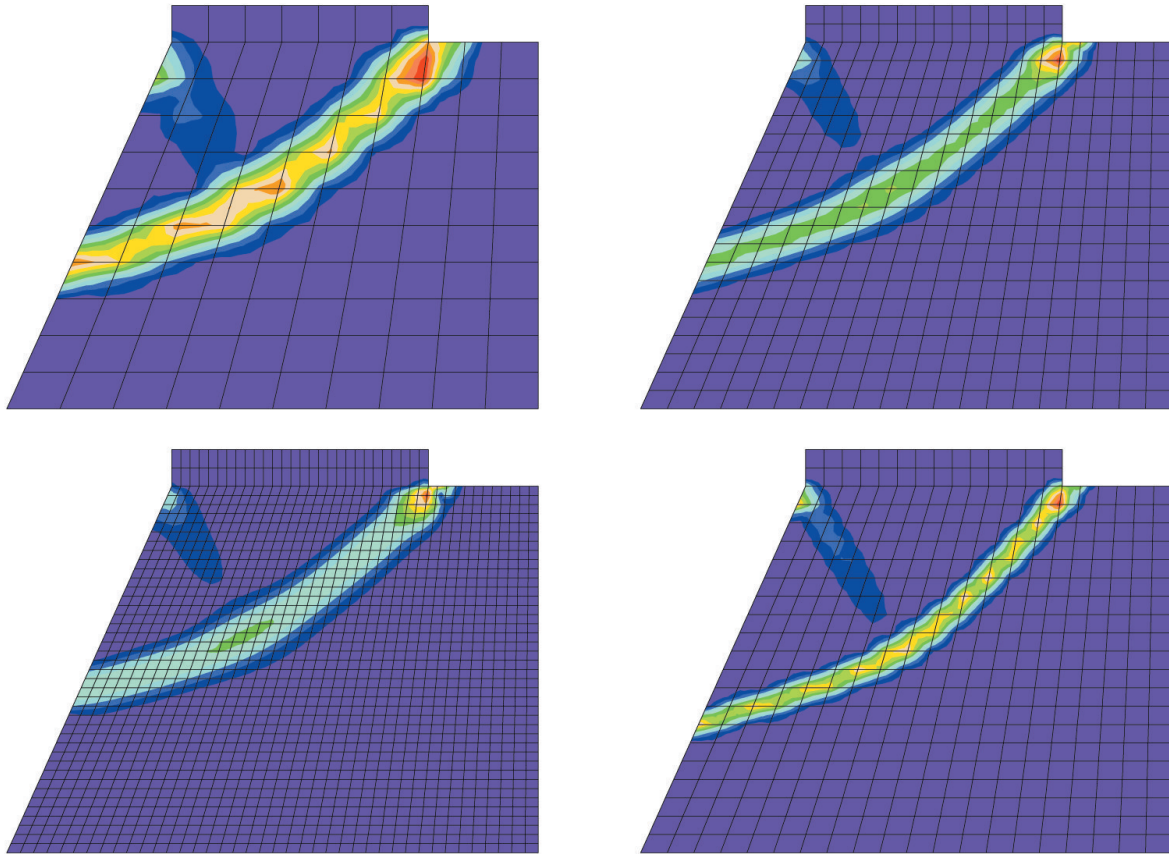


Fig. 5. Simulation of plastic deformation localized in shear bands using 3 mesh densities. Isolines represent distribution of equivalent plastic strain for $l = 0.4$ m and $v_{top} = 0.4$ mm, with the exception of lower right-hand figure for which $l = 0.2$ m and $v_{top} = 0.3$ mm

In Figure 4 the relation between the sum of forces applied to the stamp and the vertical displacement of the left node on the contact line stamp-slope. The diagrams for $l = 0.4$ m and different meshes show only small differences. In Figure 5 the simulated equivalent plastic strain distributions for the three finite element meshes are presented.

For a sufficiently dense discretization the solutions are in good agreement (the shear band width is almost the same) and they simulate the slope sliding phenomenon in a physically convincing manner. For a two times smaller parameter $l = 0.2$ m the response of the configuration exhibits smaller load-carrying capacity and stronger softening, since localization takes place in a narrower shear band. This also means that a model which is equipped with an internal length scale makes size effect simulations possible.

In turn, the gradient damage theory comes from Peerlings, de Borst and Geers, see e.g. (Peerlings *et al.* 1996; Geers 1997). It turned out that the so-called implicit gradient model is especially convenient. In this model the equilibrium equations are coupled to an averaging differential equation for a strain measure at a point. The averaged strain measure is the additional fundamental unknown which is discreti-

zed so that two-field finite elements ensue. As shown in (Peerlings *et al.* 2001), for the implicit models there are equivalent nonlocal integral models, since all higher order gradients are represented in the averaging equation.

As an example of gradient damage application a half of a beam in four-point bending is considered, as shown in Figure 6, see also (Pamin 2004; Wosatko 2008). The beam has only main longitudinal reinforcement and no stirrups. The model of concrete is specified as follows: Young's modulus $E_c = 28\,850$ N/mm², Poisson's ratio $\nu = 0.2$, modified von Mises definition of equivalent strain measure with damage threshold $\kappa_o = f_t/E_c = 0.000096$, exponential damage growth function according to (Mazars and Pijaudier-Cabot 1989) with parameters $\alpha = 0.99$, $\eta = 600$, and gradient influence factor $c = 0.5l^2 = 20$ mm². For the reinforcing steel the parameters are: Young's modulus $E_s = 210\,000$ N/mm² and yield strength $\sigma_y = 440$ N/mm². Elastic bond-slip relation is assumed.

The crack pattern presented in Figure 7 was observed in a laboratory experiment (Walraven 1978). The figure also shows the evolution of the averaged (nonlocal) strain measure simulated using the gradient damage model.

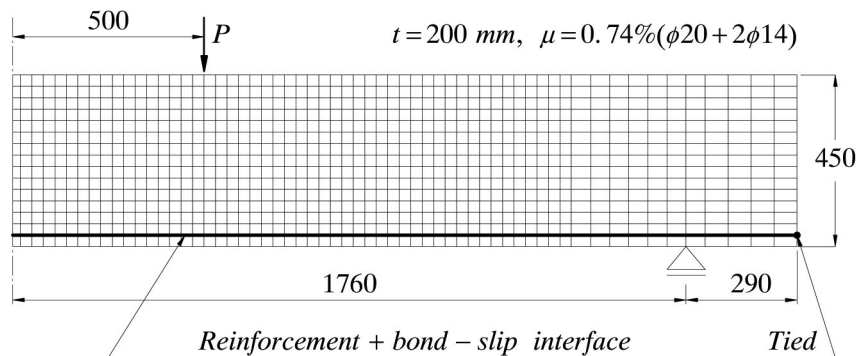


Fig. 6. Finite element model of a half of beam subjected to four-point bending

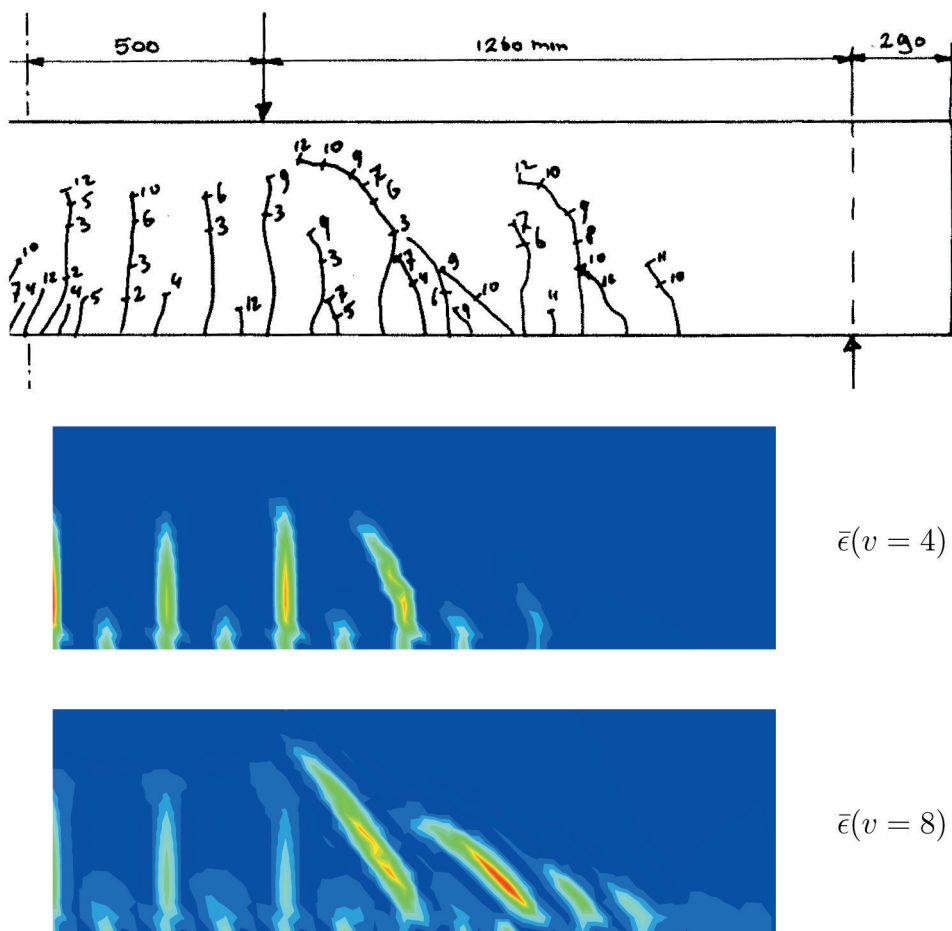


Fig. 7. Experimental crack distribution (Walraven 1978) and simulated distribution of averaged strain measure $\bar{\epsilon}$ for two values of vertical displacement v at the point where the force is applied

5. FINAL REMARKS

The application of regularized continuum models (e.g. gradient-enhanced ones) in the simulation of localization phenomena has mathematical, numerical and physical motivation. They are applied in order to:

- preserve the well-posedness of (I)BVP for softening material models,
- eliminate the pathological mesh-sensitivity of solutions obtained using discretization methods,

- represent microstructure evolution within a simplified phenomenological (macroscopic) description,
- facilitate a correct simulation of various observed phenomena: localized deformation, fracture and failure, size effects, crack or shear band patterns.

To complete this short presentation it is emphasized that obviously discretization itself does not regularize the mathematical model. Therefore, an internal length scale is necessary in the continuum description. In the absence of length scale the node spacing in a mesh determines the smallest

width of localization zone admitted by a discrete numerical model.

It seems justified to express the opinion that the knowledge and implementation of higher order continuum models are mature enough for them to be applied in the analyses of structures made from softening materials. At small strain these are quasi-brittle materials (e.g. concrete) and frictional materials (e.g. soil), but at large strain also various composites and metal alloys exhibiting induced damage. In particular, it seems that the most realistic computational approach to fracture modelling is a combination of regularized continuum description with a discontinuity representation by the XFEM, see e.g. (Simone *et al.* 2003).

It is also mentioned that an important domain of application of regularized continuum models lies in the integration of scales where one would like to model softening of a heterogeneous material phase at meso- or microscale, while this phenomenon makes the use of homogenization concepts questionable, cf. (Gitman 2006; Geers *et al.* 2010).

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