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The optimal portfolio in respect to Expected Shortfall: a comparative study

1. Introduction

Dependence structures in capital markets have recently attracted increasing attention among economists, empirical researchers, and practitioners. In order to control a portfolio for risks, portfolio managers and regulators have to take into account a degree of dependence between international equity markets when studying returns across international financial markets. Therefore, the topic of asymmetric dependence structures, such as high dependence in a bear period of the stock market is very important for both the risk control and the policy management. In addition, the benefits derived from an international diversification of asset allocation are often affected by asymmetric dependence structures.

It is well known and widely discussed in the literature that linkages among international capital markets are mostly asymmetric. From this asymmetry researchers draw a conclusion that in a bear phase, returns tend to be more interrelated than they are in a bull phase of capital markets. From this observation serious theoretical consequences for an international portfolio follow. The most important implication is a possible loss of diversification benefits in a bear time due to the rise in the dependence among capital markets. In other words, international portfolios become much more risky in bad times of stock markets that assumed in good times. The observed asymmetric dependence is an essential source of rise in the costs of a diversification with foreign equities.

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In this article we investigate how model selection affects the calculated risk of financial position. The two standard models are mean-variance Markowitz model and multivariate GARCH model. Both models assume symmetric and thin-tailed distributions of returns, in particular they assume the normal distribution. Recently developed models based on copula functions are both flexible and convenient to model anomalies in distributions, such as an asymmetry or fat-tails. In this article we focus on regime switching copula models. We consider two risk measures: Value at Risk and Expected Shortfall. The expected risk derived on the basis of the regime switching copula model is compared to the expect risks obtained by using the Markowitz model and the multidimensional GARCH model.

A model misspecification may cause a number of problems. Incorrect evaluation of the expected value of a financial position is one of the most serious drawbacks of the financial models. However, a risk underestimation may cause even worse consequences. Most of risk measures are strongly, or entirely, dependent on distributions of tails. Especially, the dependence of extreme assets' values substantially affects the distribution of the portfolio value. Therefore, an omission of an asymmetry or a high kurtosis of assets' distributions may be a reason for a miscalculation of risks.

The remainder of the contribution is organized in the following way: in section 2 we conduct the literature overview concerning the dependence concepts, including regime switching models and copulas and discuss the recent contributions to the subject; in section 3 the dependence measures and copulas are overviewed; in the following section the copula regime switching model is described; in the fifth section risk measures based on copula models are discussed; in the sixth section we present the data, report and discuss the results; section 7 concludes the paper.

2. Literature overview

Relations among international stock markets have been investigated in many papers, especially in the period of the financial crises. The topic under study is important for market participants, because, due to the globalization process, the global markets are becoming more and more dependent. This observation follows from the liberalization and deregulations in both money and capital markets. In addition, the globalization process diminishes opportunities for international diversification.

Numerous recent studies deal with an asymmetry in dependence structures in international stock markets. They reveal two interesting asymmetries. The dependence tends to be high in both highly volatile markets and bear markets.

While in some studies, the evidence of the first type of asymmetry is shown, several other studies found the second asymmetry. In one of the earliest contribution, Hamao, Masulis, and Ng [18] investigated the relations among equity markets across Japan, the U.K., and the U.S. using the daily data of stock indices. The authors estimated the GARCH-M model. Using this model the authors established volatility spillover effects from the U.S. and U.K. stock markets to the Japanese market. King and Wadhwani [23] developed a contagion mechanism model. They detected contagion effects. The contributors stressed that an increase in volatility by using a high frequency data from the stock markets in Japan, the U.K., and the U.S strengthened these effects. These findings were supported to some extent by Lin et al. [26] who analysed two international transmission mechanism models based on the daily returns of stock indices in Japan and the U.S. Erb et al. [14] found that monthly cross-equity correlations among the G7 countries were highest when any of two countries were in a recession. In addition, the contributors claimed that they are much higher in bear markets. In the paper by Longin and Solnik [27], the monthly data of stock indices for several industrial countries were analyzed. The contributors, using a multivariate GARCH model, found that the correlations between major stock markets raised in periods of a high volatility. On a basis of the multivariate SWARCH model, Ramchand and Susmel [36] found that monthly returns of stock markets in the U.K., Germany, and Canada tended to be essentially more correlated with the U.S. equity market during periods of a high U.S. market volatility. The similar results could be found in King, Sentana, and Wadhwani [22], Ball and Torous [5], Bekaert and Wu [6], Ang and Bekaert [2], and Das and Uppal [10].

Following Davison and Smith [11] and Ledford and Tawn [25], Longin and Solnik [28] derived a method to measure the extreme high correlation by the conditional tail correlation based on extreme value theory. The contributors established that the conditional correlation between the U.S. and other G5 countries strongly increases in bear markets. In contrary, the conditional correlation does not essentially increase in bull markets.

In more recent studies by Campbell et al. [7], Ang and Bekaert[2], Das and Uppal [10], Patton [34], and Poon et al. [35], the existence of two regimes in international equity markets was suggested: a high dependence regime with low and volatile returns and a low dependence regime with high and stable returns.

Based on this hypothesis, Ang and Bekaert [2] estimated a Markov switching multivariate normal (MSMVN) model using the U.S., the U.K., and German monthly stock indices. The contributors detected some evidence that a bear regime is characterized by low expected returns, high volatility, and high correlation, whereas a normal regime is characterized by high expected returns, low volatility, and low correlation. Their model was able to replicate Longin and

Solnik's [28] results. Referring to Ang and Chen [2], they demonstrated that an asymmetric bivariate GARCH model, widely used in the literature to analyze the international stock markets, cannot replicate them.

In recent times, copulas have become a major tool in the finance for modelling and analyzing dependence structures between financial variables. In contrast to the linear correlation, the copula reflects the complete dependence structure inherent in a random vector (see [13]). In finance, copulas have attracted much attention in the calculation of the Value-at-Risk (VaR) of market portfolios (see e.g. Junker and May, 2005; Kole et al., 2007 and Malevergne and Sornette, 2003) and the modelling of the credit default risk.

Ball and Torous [5] and Guidolin and Timmermann [17] investigated the economic significance of their empirical findings from a risk management point of view. Rodriguez [37] used copula model with Markov switching parameters. Okimoto [32] stressed that ignoring the asymmetry in bear markets could be costly when risk measures are evaluated. In his contribution, using a copula based regime switching Markov model, he concentrated on the value at risk (VaR) and expected shortfall (ES).

According to his calculation, ignoring such an asymmetry in bear markets significantly affects risk measures, i.e. the 99% VaR is undervalued by about 10%, while the expected shortfall is undervalued by about 5% to 10% consistently over the whole significance level between 90% to 99%. This is essential for the risk management.

The empirical literature on the optimal choice of the parametric copula family for the VaR-estimation can be clustered into three groups.

The first group of contributors claims that the elliptical copulas are optimal. The representative of this stream of papers is e.g. paper by Malevergne and Sornette [29]. This is one of the first empirical studies on the optimality of copula models for the modelling of dependence structures of linear assets. The authors, based on the dataset consisting of six FX futures, six commodity prices and 22 stocks listed on the NYSE, demonstrated that the dependence structures of the majority of bivariate portfolios built from these assets can be correctly reflected by a Gaussian copula. However, in the opinion of the contributors, their result can be biased. The reason is that Student's t copula can easily be mistaken for a Gaussian copula. In addition, Malevergne and Sornette [29] did not include the estimation of a risk measure or Goodness of Fit –tests (abbreviation GoF-tests). Kole et al. [24] found, on the basis of just one trivariate portfolio (one stock-, one bond- and one REITS-index), that the Student's t copula is the best for modelling the dependence structure of linear assets. DiClemente and Romano [12] using the 20-dimensional portfolio of Italian stocks, demonstrated that a model incorporating margins following an extreme value distribution and an elliptical copula can yield much better VaR-estimates than the classical correlation-based model. However, they used neither Archimedean copulas nor copula-GoF- tests. In contribution by Fantazzini [15], it is shown that three bivariate portfolios built from stock indices can be well modelled by a constant or dynamic Gaussian copula in order to estimate VaR properly.

The second stream of studies justifies an optimality of Archimedean copulas. Junker and May [21] argued that a transformed Frank copula with GARCH margins can improve VaR- and ES-estimates in comparison to elliptical copula models. However, their conclusions are based solely on the single bivariate portfolio of German stocks. In addition, they only apply GoF-tests for general distributions. They were not adjusted to the characteristics of copulas. Similar results were presented by Palaro and Hotta [33] for the bivariate portfolio based on the S&P 500- and the NASDAQ- index. The authors showed that a symmetrised Joe-Clayton copula joint with GARCH margins performs significantly better than elliptical copula models.

Recent studies, belonging mostly to a third cluster of research, demonstrate that the optimal parametric copula as well as the strength and structure of the dependence between asset returns are not constant over the time. In order to allow the parametric form of the copula to change over time more recent studies like the ones addressed above Rodriguez [37], Okimoto [32], Chollette, Heinen, and Valdesogo [8] and Markwat, Kole, and van Dijk, [31], Weiss [38] apply the convex combinations of copulas. The contributors drew a conclusion that more flexible mixture copula models yield better VaR and ES estimates than unconditional copula models.

The contributors stressed that copula models perform better than correlation-based models with respect to the estimation of VaR. This was the case when the optimal parametric copula family was known ex ante.

The main aim of this contribution is a comparison of the expected shortfall for returns derived on the basis of the Markowitz model, the multidimensional GARCH model and the copula regime switching model.

3. Dependence measures based on copulas

The correct evaluation of the dependence between assets' interest rates is essential for an accurate assessment of an investment risk. In the case of risk management, the dependence between negative values, in particular between extreme negative values plays a key role. Especially, if such a dependence is substantial, then an investor can lower the risk by diversification of a portfolio to less than expected. In this section we present some functions measuring the dependence between random variables and discuss their intuitive meaning. Moreover, we describe the presented dependence measures' relationship with copulas.

3.1. Exceedance correlation coefficient

The most traditional dependence measure is Pearson correlation. However, it measures only linear dependence and works only in the range of the spherical and elliptical distributions. The exceedance correlation is the generalized Pearson coefficient which measures asymmetric dependence. It is defined as the correlation between two variables, conditional on both variables being below or above some fixed levels. Exceedance correlation coefficients between random variables X and Y are defined as:

$$ecorr_{\theta_1\theta_2}^{L}(X,Y) := corr(X,Y|X \leq \theta_1, Y \leq \theta_2), \quad (1)$$

$$ecorr_{\theta_1\theta_2}^{U}(X,Y) := corr(X,Y|X \geq \theta_1, Y \geq \theta_2), \quad (2)$$

where $ecorr_{\theta_1\theta_2}^{L}$ is lower exceedence correlation, $ecorr_{\theta_1\theta_2}^{U}$ is upper exceedence correlation and θ_1, θ_2 are fixed thresholds.

Properly calculated exceedance correlation would be an efficient tool in risk management, where negative extreme values of an investment return are crucial. However, this coefficient has some drawbacks. For instance, it is computed only from observations which are below (above) the fixed limit. Therefore, as the limit is further out into the tail as exceedance correlation is computed less precisely. Another inconvenience with the exceedance correlation is that it is dependent on margins, thus it cannot be calculated only from the copula connecting variables.

3.2. Tail dependence

Another tail dependence measure is quantile dependence. For random variables X and Y with distribution functions F and G, respectively, the lower tail dependence at threshold α is defined as $P\left[Y < G^{-1}(\alpha)|X < F^{-1}(\alpha)\right]$. Analogously, the upper tail dependence at threshold α is defined as $P\left[Y > G^{-1}(\alpha)|X > F^{-1}(\alpha)\right]$. The dependence measure which is particularly interesting is the tail dependence obtained as the limit of a quantile dependence. We define lower tail dependence λ_L of X and Y as:

$$\lambda_L = \lim_{\alpha \to 0^+} P\left[Y < G^{-1}(\alpha)|X < F^{-1}(\alpha)\right], \quad (3)$$

and upper tail dependence Λ_u of X and Y as:

$$\lambda_U = \lim_{\alpha \to 1^-} P\left[Y > G^{-1}(\alpha)|X > F^{-1}(\alpha)\right]. \quad (4)$$

Variables X and Y are called asymptotically dependent if $\lambda_L \in (0,1]$ and asymptotically independent if $\lambda_L = 0$ For variables connected by the copula C, lower tail dependence λ_L and upper tail dependence λ_U can be computed as follows:

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}, \tag{5}$$

$$\lambda_U = \lim_{u \to 1^-} \frac{\bar{C}(u,u)}{1-u}, \tag{6}$$

where $\bar{C}$ is the survival copula defined by:

$$\bar{C}(u,v) = C(1-u, 1-v) - u - v + 1u, \text{ for } u,u \in (0,1] \tag{7}$$

Unlike exceedance correlations, tail dependence is independent of margins. In the most cases, for a given copula, one can simply calculate tail dependences using formulas (5) and (6). In Table 1, we present results for the copulas used in the paper.

Table 1

Tail dependencies for Gaussian, BB1, BB4, BB7 copulas

	λ_L	λ_U
C_ρ^{Gauss}	0	0
$C_{\theta,\delta}^{BB1}$	$2^{-\frac{1}{\delta\theta}}$	$2-2^{\frac{1}{\delta}}$
$C_{\theta,\delta}^{BB4}$	$\left(2-2^{-\frac{1}{\delta}}\right)^{-\frac{1}{\theta}}$	$2^{-\frac{1}{\delta}}$
$C_{\theta,\delta}^{BB7}$	$2^{-\frac{1}{\delta}}$	$2-2^{\frac{1}{\theta}}$

3.3. Kendall's τ

Another class of dependence measures is based on ranks of variables. The two most popular rank correlations coefficients are Kendall's τ and Spearman's ρ. Both rely on the notion of the concordance. Let (x_1, y_1) and (x_2, y_2) be two observations of the random vector (X, Y). We say that the pair is *concordant* whenever $(y_1 - y_2)(x_1 - x_2) > 0$, and *discordant* whenever $(y_1 - y_2)(x_1 - x_2) < 0$. Intuitively, a pair of random variables are concordant if large values of one variable occur more likely with large values of the other variable.

For random variables X and Y, Kendall's τ is defined as:

$$\tau = P\left[(y_1 - y_2)(x_1 - x_2) > 0\right] - P\left[(y_1 - y_2)(x_1 - x_2) < 0\right],$$

where (x_1, y_1) and (x_2, y_2) are independent observations of (X, Y). In terms of copulas, Kendall's τ has concise form. For the pair of random variables X and Y and its copula C, we have:

$$\tau_C = 4 \int_{[0,1]^2} C(u,v)\,dC(u,v) - 1. \tag{8}$$

Since copula is invariant with respect to any monotonic transformation, Kendall's τ has also this property. From the formula (8) we see that Kendall's τ does not depend on marginal distributions.

4. Compared models

In this section we present the regime switching copula model with GARCH margins and the estimation procedure. Other models used in this article are: the Markowitz model and *multivariate Generalized Autoregressive Conditional Heteroscedasticity* (mGARCH) model.

The Markowitz model is a standard model introduced by Markowitz. This model is based on a normal distribution assumption and does not include any dynamic changes. There are numerous papers stressing the inadequacy of this model. We believe that there are still individuals using this method. Thus, we decided to compare this method to other in the context of our study. Markowitz model's parameters can be equivalently estimated using the likelihood function maximization or the least square method.

Switching models were introduced by Hamilton [19] and widely analyzed by Hamilton [20]. Let $y_t = (y_{1t}, y_{2t})$ be a pair of interest rates of analyzed indices, and let $Y_t = (y_t, y_{t-1}, y_{t-2}, \ldots)$ be the series of observations available at the time t.

We denote the two-state Markov state process by s_t, which has two possible values, say 1 and 2, we call these states regimes. We choose the first regime copula from copulas with non-zero tail dependencies, namely BB1, BB4 and BB7 copulas. The second copula is the Gaussian copula, which corresponds to symmetry and tail independence of the investigated variables.

The conditional joint density function f for y_t is defined as:

$$f(y_t | Y_{t-1}, s_t = j) = c^{(j)}\left(F_1(y_{1t};\delta_1), F_2(y_{2t};\delta_2)\right) \cdot f_1(y_{1t};\delta_1) \cdot f_2(y_{2t};\delta_2), \tag{9}$$

where F_i and f_i, for $i = 1,2$, are the marginal distribution functions and density functions of corresponding variables, and δ_i is a parameter vector for the marginal distribution. The probability that the state i precedes the state j is denoted by $p_{ij} = P[s_t = j | s_{t-1} = i]$.

All four probabilities form transition matrix:

$$\boldsymbol{P}=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}=\begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}. \tag{10}$$

The estimation of the regime switching copula model is based on the maximum likelihood estimation. Unfortunately, the computing power needed to maximize likelihood function is enormous. To simplify the calculation, the decomposition of likelihood function to margins likelihood functions and the dependence likelihood function is performed. Formally, for a given sample $Y=(Y_1, Y_2,\ldots, Y_T)$, the log-likelihood function is defined by:

$$L(\boldsymbol{Y};\delta,\theta)=\sum_{t=1}^{T}\ln f(y_t \mid Y_{t-1};\delta,\theta),$$

and it is decomposed to L_m and L_c such that:

$$L(\boldsymbol{Y};\delta,\theta)=L_m(\boldsymbol{Y};\delta)+L_c(\boldsymbol{Y};\delta,\theta),$$

where:

$$L_m(\boldsymbol{Y};\delta)=\sum_{t=1}^{T}\Big[\ln f_1\big(y_{1t} \mid (Y_{1t-1};\delta_1)\big)+\ln f_2\big(y_{2t} \big| (Y_{2t-1};\delta_2)\big)\Big], \tag{11}$$

$$L_c(\boldsymbol{Y};\delta,\theta)=\sum_{t=1}^{T}\ln c[F_1\big(y_{1t} \mid (Y_{1t-1};\delta_1\big),F_2\big(y_{2t} \big| (Y_{2t-1};\delta_2\big);\theta\big]. \tag{12}$$

The parameters of the model are estimated as follows. In the first step we estimate the parameters δ_1 and δ_2 of the marginal distribution. This step is performed by the maximization of the likelihood function defined by (11). In the second step we maximize the likelihood function defined by (12) to estimate parameters θ_1 and θ_2 of copulas $c^{(1)}$and $c^{(2)}$, and transition matrix given by (10). Note that parameters δ_1, δ_2, θ are in fact collections of parameters.

A method of the estimation of marginal distributions depends on the model which is chosen to describe the specific marginal variable. To model the mean of a time series, we use the simple autoregressive model. As we mentioned before, usually for time series of returns hypotheses of normal distribution of residuals are rejected. In particular, investigated time series are fat-tailed, asymmetric and heteroscedastic. Therefore, for every analyzed time series r_t, we use the following AR(1)-GARCH(1,1) model:

$$r_t=\varphi_0+\varphi_1 r_{t-1}+\varepsilon_t \tag{13}$$

$$h_t=\omega+\alpha\varepsilon_{t-1}^2+\beta h_{t-1} \text{ for } \omega>0,\ \alpha\geq 0,\ \beta\geq 0; \tag{14}$$

where $\varepsilon_t = h_t e_t$ and e_t is a white noise. Although, with respect to an asymmetry and a fat tail, e_t is described by the skewed Student-t distribution. The skewed

Student-*t* is a two parameter distribution. For v > 2 and λ ∈ [–1,1], the skewed Student-*t* density function, denoted by *St*, is defined by:

$$St_{\nu,\lambda}(x)=\begin{cases} bc\left(1+\frac{1}{\nu-2}\left(\frac{bx+a}{1-\lambda}\right)^2\right)^{\frac{-(\nu+1)}{2}} & \text{dla}\, x<-\frac{a}{b} \\ bc\left(1+\frac{1}{\nu-2}\left(\frac{bx+a}{1+\lambda}\right)^2\right)^{\frac{-(\nu+1)}{2}} & \text{dla}\, x\geq-\frac{a}{b}\end{cases}, \tag{15}$$

where $a=4\lambda c\left(\frac{\nu-2}{v-1}\right), b=\sqrt{1+3\lambda^2-a^2}, c=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}$,

The second step is the estimation of copulas parameters and transition probabilities. To do so, we use Hamilton filter. For the transition matrix $\boldsymbol{P}$ given by (10), we define:

$$\hat{\xi}_{t|t}=\frac{\hat{\xi}_{t|t-1}\odot\eta_t}{\mathbf{1}^T(\hat{\xi}_{t|t-1}\odot\eta_t)}, \tag{16}$$

$$\hat{\xi}_{t+1|t}=\boldsymbol{P}^T\hat{\xi}_{t|t}, \tag{17}$$

where $\hat{\xi}_{t|t}=P[s_t=j\,|\,Y_t;\theta]$ and $\hat{\xi}_{t+1|t}=P[s_{t+1}=j|Y_t;\theta]$ the Hadamard's multiplication denoted by ⊙ means the multiplication coordinate by coordinate. The vector of copulas' densities is denoted by η_t,

$$\eta_t=\begin{bmatrix} c^{(1)}\left(F_1(y_{1t};\delta_1),F_2\left(y_{2t};\delta_2\right);\theta_1\right) \\ c^{(2)}\left(F_1(y_{1t};\delta_1),F_2\left(y_{2t};\delta_2\right);\theta_2\right)\end{bmatrix}. \tag{18}$$

The log-likelihood function defined by (12) for the observed data can be written as:

$$L_c\left(\boldsymbol{Y};\delta,\theta\right)=\sum_{t=1}^{T}\ln\left(\mathbf{1}^T\left(\hat{\xi}_{t|t-1}\odot\eta_t\right)\right), \tag{19}$$

where the initial value $\hat{\xi}_{1|0}$ is the limit probability vector:

$$\hat{\xi}_{1|0}=\begin{bmatrix}\frac{1-p_{22}}{2-p_{11}-p_{22}} \\ \frac{1-p_{11}}{2-p_{11}-p_{22}}\end{bmatrix} \tag{20}$$

Models based on mGARCH have been recently broadly used and modified. In this article, conditional mean dynamics is described by the VAR(1) model. For details of the recent study we refer to Croux and Joossens [9]. To model conditional correlation, we use the Dynamic Conditional Correlation (DCC) model with normal conditional distributions.

Under this model the conditional mean of the multidimensional time series y at the time t is computed as follows:

$$\mathrm{E}[y_t \mid \Omega_{t-1}] = \mu + Ay_{t-1} + \varepsilon_t, \tag{21}$$

where μ is constant, Ω_t is the information set available at the time t and A is a vector autoregressive matrix. The error term ε_t at the time t is defined by:

$$\varepsilon_t = H_t^{(1/2)} z_t, \tag{22}$$

where z_t is a sequence of N- dimensional, in our case $N = 2$, i.i.d. random vector with the following characteristics: $E(z_t) = 0$ and $E\left(z_t z_t^T\right) = I_N$, therefore $z_t \sim N(0, I_N)$. The dynamic covariance matrix H_t is decomposed to:

$$H_t = D_t R_t D_t, \tag{23}$$

where D_t^2 is a dynamic variance matrix and R_t is a dynamic correlation matrix. In the two-dimensional case, $D_t = \mathrm{diag}\left(\sqrt{\boldsymbol{h}_{11,t}}, \sqrt{\boldsymbol{h}_{22,t}}\right)$, where

$$\boldsymbol{h}_t = \omega + \alpha\varepsilon_{t-1} \odot \varepsilon_{t-1} + \beta\boldsymbol{h}_{t-1}. \tag{24}$$

The correlation matrix R_t is decomposed as follows:

$$R_t = \left\{\mathrm{diag}(Q_t)\right\}^{-\frac{1}{2}} \cdot Q_t \cdot \left\{\mathrm{diag}(Q_t)\right\}^{-\frac{1}{2}}. \tag{25}$$

The correlation driving process Q_t is defined by:

$$Q_t = \left(1 - \alpha - \beta\right)\bar{Q} + \alpha P_{t-1}^{*} + \beta Q_{t-1}, \tag{26}$$

where $\bar{Q}$ denotes unconditional correlation matrix of the stantarized errors and

$$P_t^{*} = \left\{\mathrm{diag}(Q_t)\right\}^{-\frac{1}{2}} \cdot D_t^{-1} \cdot Q_t \cdot D_t^{-1} \cdot \left\{\mathrm{diag}(Q_t)\right\}^{-\frac{1}{2}}. \tag{27}$$

This particular specification of the DCC model has been proposed by Ailelli [1].

5. Portfolio optimization

The portfolio optimization problem is widely analyzed. There are two main goals to achieve in any portfolio optimization problem. The first aim is the maximization of the expected value of the portfolio. The most natural way is to maximize the expected nominal value, a generalization of this approach is the maximization of an expected utility. In this article, we do not consider utility functions, for more details about a maximizing an expected utility see Föllmer and Schied [16]. The second aim in the portfolio optimization is to minimize a risk. There are numerous approaches to a concept of risk. The most standard understanding of a risk is an uncertainty. For any portfolio, its risk may be understood as the variance of the future value of the portfolio. This concept was firstly introduced in [30] and the corresponding portfolio optimization problem was solved in this paper.

In this article, we deal with the concept of risk proposed in [4]. We analyze the risks of the financial positions in the one period case. It means that the value of the financial position under study in the end of the period turns into a random variable.

The function $\rho\colon \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ is the family of all attainable financial positions, is called *risk measure* if it satisfies the following properties for all financial positions X, Y:

1. *Monotonicity:*
   If $X \leq Y$, then $\rho(X) \geq \rho(Y)$. (28)
2. *Cash invariance:*
   If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$. (29)

The interpretation of monotonicity is clear: The increase of a financial position's payoff profile do not increase its risk. The cash invariance is motivated by the interpretation of $\rho(X)$ as a capital requirement. If the amount m is added to the position and invested in a risk-free manner, the capital requirement is reduced by the same amount.

It is usually assumed that the portfolio diversification should not increase the risk. *Convex risk measures* has this property, the risk measure ρ is called *convex risk measure* if it satisfies the following convexity property for all financial positions X, Y:

$$\rho\left(\lambda X + (1-\lambda)Y\right) \leq \lambda\rho\left(X\right) + (1-\lambda)\rho(Y), \text{ , for all } 0 \leq \lambda \leq 1. \tag{30}$$

Moreover the convex risk measure ρ is called *coherent risk measure* if it satisfies the following positive homogeneous property:

$$\rho\left(\lambda X\right) \leq \lambda\rho\left(X\right), \text{ , for all } 0 \leq \lambda \text{ and } X \in \mathcal{X}. \tag{31}$$

Value at Risk (VaR) is an approach to the problem of measuring the risk of a financial position X based on specifying a quantile of the distribution of X under the given probability measure. Value at Risk is the smallest amount of capital which, if added to X and invested in the risk-free asset, keeps the probability of a negative outcome below some fixed level.

For $X \in \mathcal{X}$ and $\lambda \in (0,1)$ we define *Value at Risk at level* λ as:

$$VaR_{\lambda}(X) := \inf\{m | \mathrm{P}[X + m < 0] \le \lambda\}. \tag{32}$$

In the other words, $VaR_{\lambda}(X)$ is $(1 - \lambda)$-quantile of the variable $(-X)$. Clearly, *VaR* is a positively homogeneous risk measure. Generally, Value at Risk is not a convex risk measure. However, it is convex if it measures a risk of financial positions come from some particular classes. For instance, *VaR* is convex risk measure if X consists of only normally distributed financial positions.

This risk measure has a clear interpretation and is recommended by numerous financial institutions and presented in documents such as the Basel Accords. However, the absence of the convexity is a substantial objection. This disadvantage of *VaR* led researchers to convex risk measures which have similar interpretation as Value at Risk. It appears that, so called *Expected Shortfall* (ES), is a convex risk measure.

For $X \in \mathcal{X}$ and $\lambda \in (0,1)$ we define *Expected Shortfall at level* λ as:

$$ES_{\lambda}(X) := \mathrm{E}[VaR_{\alpha} | \alpha \le \lambda] \tag{33}$$

This convex risk measure is also called *Conditional Value at Risk* (*CVaR*), *Average Value at Risk (AVaR), Tail Value at Risk (TVaR), Mean Excess Loss or Mean Shortfall*. However, there are other risk measures defined in some papers under these names. In this article, the risk measure defined by (33) is called an Expected Shortfall. Clearly, $ES_{\lambda}(X) \ge VaR_{\alpha}$, for any $\lambda \in (0,1)$.

In general case it is difficult or impossible to find an analytical form of *ES*. One can notice that there does not exist an analytical form of *VaR* for normally distributed financial positions. We estimate *VaR* using the Monte Carlo method. For every analyzed model, we simulate 1,000,000 observations. It is usually recommended to simulate 100,000 observations. However, we are mostly interested in extreme observations, namely those which are below VaR_{λ}-level. In the formula (33), one can see that ES_{λ} is determined by a conditional distribution, in particular by the financial position's distribution in the lower λ-tail.

6. The data and the estimation results

The database consists of prices of three stock market indices. Namely, the American DJIA, the German DAX and the Austrian ATX. In order to avoid introducing an artificial dependence due to the difference in closing times of stock exchanges around the globe, we work with Wednesday to Wednesday returns. Comparing to daily returns, weekly return processes have lower autocorrelation and avoid the missing data problem. This gives us a sample of 689 weekly returns from January 2000 to March 2013. We apply continuous (logarithmic) returns:

$$r_t = 100 \cdot \log \frac{p_t}{p_{t-1}}, \tag{34}$$

where p_t is the price index at the time t.

Firstly, we present some descriptive statistics in Table 2.

Table 2

Logarithmic rates of return time series summary statistics

	ATX	DAX	DJIA
Mean	0.1036	0.0248	0.0335
Median	0.4157	0.3984	0.2140
Std. dev.	3.4646	3.4630	2.5782
Kurtosis	16.7931	5.1127	7.7125
Skewness	–1.9245	–0.6643	–0.9464

In the period under study we observe an insignificant positive means in all the three indices. A relatively large absolute value of median suggest asymmetries in the examined time series. Negative skewnesses confirm this conjecture. These asymmetries suggest that normal distribution should not be used to model these time series, and high kurtosis in all the three time series confirms that.

Table 3 presents empirical dependence measures for analyzed pairs of price indices.

Table 3

Empirical dependences between price indices' time series

	ATX/DAX	ATX/DJIA	DAX/DJIA
ρ	0,6439	0.6056	0,7863
Kendall's τ	0.4135	0.3704	0.5848
$\boldsymbol{\lambda_L}$	0.6421	0.5072	0.5797
$\boldsymbol{\lambda_U}$	0.4638	0.3623	0.5652
$\boldsymbol{ecorr^L_{Q_1^1,Q_1^2}}$	0.7179	0.6441	0.6781
$\boldsymbol{ecorr^U_{Q_3^1,Q_3^2}}$	0.3798	0.5302	0.7183

Here ρ is Pearson's correlation, Q_α^1 and Q_α^2 are α-quartiles of a realized volatility series and a daily volume series, respectively. Tail dependencies λ_L and λ_U are approximated by $P\left[Y < G^{-1}(0.1) | X < F^{-1}(0.1)\right]$ and $P\left[Y > G^{-1}(0.9) | X > F^{-1}(0.9)\right]$, respectively.

One can observe the strong and significant linear correlation between the indices under consideration. As expected, the strongest dependence is observed for the DAX/DJIA pair. Despite the many drawbacks of linear correlation, it is worth to mention that a portfolio construction is very sensitive to the degree of dependence.

Asymmetries in tails are observed for the ATX/DAX and ATX/DJIA pair. For the DAX/DJIA pair, the lower and the upper estimated tail dependence are at similar levels. The same result is observed for exceedence correlations.

A multidimensional GARCH(1,1) model with conditional mean described by the VAR(1) is supposed to eliminate the incorrect assessments of the foregoing model. Table 4 presents A matrices and constants μ from equation (21) for the three pairs of analysed time series:

Table 4

Vector autoregressive parameters

	ATX	**DAX**		**ATX**	**DJIA**		**DAX**	**DJIA**
ATX	–0.0738	0.0855	**ATX**	–0.1274	0.2409	**DAX**	–0.0937	0.0880
DAX	–0.0640	–0.0009	**DJIA**	–0.0006	–0.0754	**DJIA**	–0.0265	–0.0479
$\boldsymbol{\mu}$	0.1118	0.0235	$\boldsymbol{\mu}$	0.1117	0.0334	$\boldsymbol{\mu}$	0.0160	0.0333

Estimated parameters of GARCH(1.1) model. described by (24) and (26). are presented in Table 5:

Table 5

Multidimensional GARCH model parameters

	ω	**α**	**β**
ATX	0.4952	0,2281	0,7467
DAX	1,2810	0,3028	0,6133
DCC		0,0363	0.9513
	ω	**α**	**β**
ATX	0.5308	0.2278	0.7411
DJIA	0.5657	0.2451	0.6830
DCC		0.0315	0.9607

Table 5 cont.

	ω	α	β
DAX	1.2065	0.2986	0.6239
DJIA	0.5601	0.2505	0.6802
DCC		0.0495	0.8987

Using methods described in section 3 we conducted the estimation of parameters of models for margins and regime-switching copulas. Table 6 contains the estimation results of AR(1)-GARCH(1.1) models along with Skeweed-*t* distributions.

Table 6

Estimation results of models for margins

parameter	φ_0	φ_1	ω	α	β	υ	λ
ATX	0.2868	–0.0267	0.4007	0.126	0.8315	–0.2211	7.5306
DAX	0.2616	–0.1133	0.5833	0.1871	0.7703	–0.3183	9.4504
DJIA	0.172	–0.1215	0.2738	0.1455	0.8127	–0.2332	7.7701

The estimated results confirm the stylized facts about log-returns: the skewness and the fat-tailedness. All of the estimated parameters are significant (5% level) with one exception (the AR(1) term in the ATX model).

We tested the correctness of the specification using the Ljung-Box and Engle tests applied to standardized residuals which are transformed to the uniform using the estimated Skewed-t distributions. Through goodness of fit tests along with the BDS test (Brock-Dechert-Scheinkman) we were able to check the uniform distribution of standardized residuals.

In the next step we estimated the regime switching copulas. To describe a dependence asymmetry we use two-parameter Archimedean copulas (BB1, BB4 and BB7) and Gaussian copula to model symmetric dependence with tail-independence patterns. In Table 7 we present the estimation results.

Table 7

Estimation results of regime switching copulas

pair of indices	first regime copula	$\theta_1^{(1)}$	$\theta_1^{(2)}$	θ_2	p_{11}	p_{22}
ATX/DAX	BB7	1.5723	1.5644	0.3430	0.9983	0.9978
DAX/DJIA	BB1	0.6434	1.8649	0.4356	0.9916	0.9246
ATX/DJIA	BB1	0.7751	1.3501	0.3561	0.9984	0.9983

All of the estimated parameters are significant. The copulas that fit the best are chosen using AIC and BIC information criterions. The correctness of the copula specification are validated by an Anderson-Darling test applied to the first derivative of copulas: $C(u|v)=\frac{dC}{du}$ and $C(v|u)=\frac{dC}{dv}$.

In addition, based on estimated parameters of the transition matrix we computed the mean time of return to regimes. In all cases this value is lower for the asymmetric regime with a dependence in tails. For all pairs, the dependence between extremely low returns is stronger than between extremely high returns. The strength of dependence measured by weighted Kendall coefficients is the strongest for the DAX/DJIA pair (with value 0.564) and the weakest for the ATX/DJIA pair (value 0.352).

The standard method of visualization of measure of risk under the assumed model is drawing of the efficient frontier line. An efficient frontier for a given measure of risk is the curve showing the minimal risk of portfolio which exhibit the calculated expected returns.

For all three indices' pairs and the two risk measures, Figures 1–6 illustrate similar relationships.

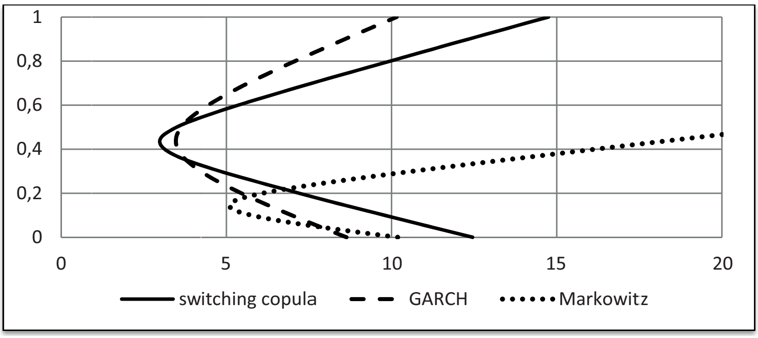

Figure 1. Efficient frontiers of Value at Risk for ATX/DAX pair

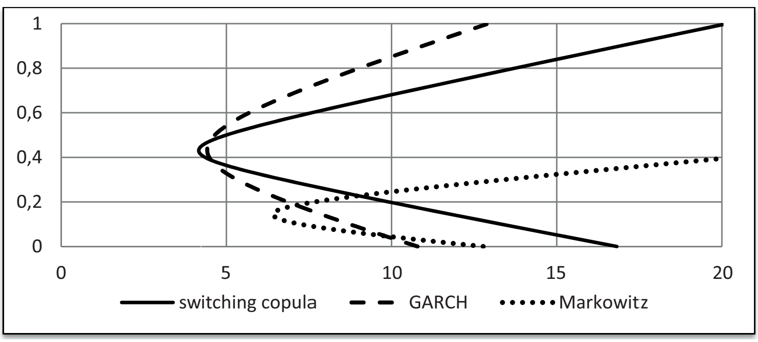

Figure 2. Efficient frontiers of Expected Shortfall for ATX/DAX pair

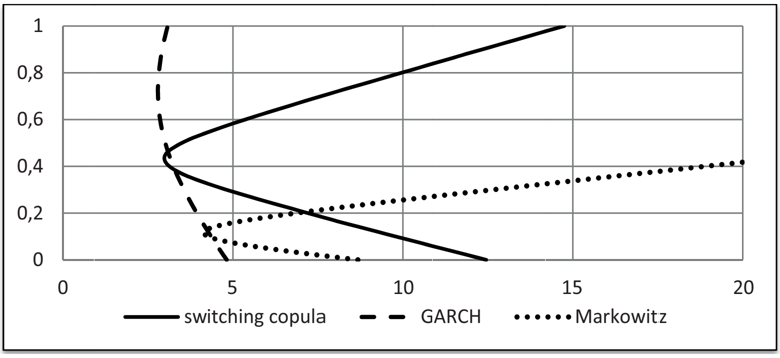

Figure 3. Efficient frontiers of Value at Risk for ATX/DJIA pair

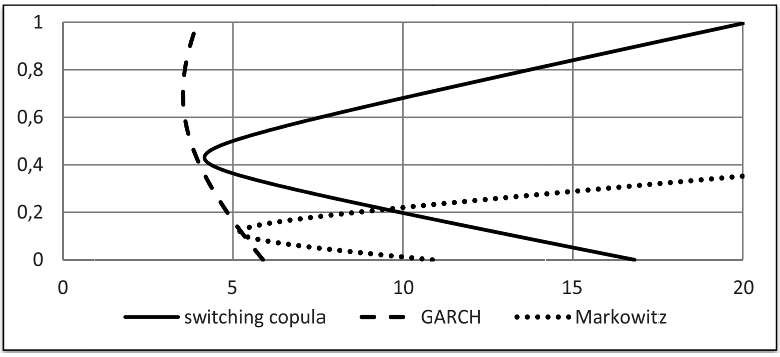

Figure 4. Efficient frontiers of Expected Shortfall for ATX/DJIA pair

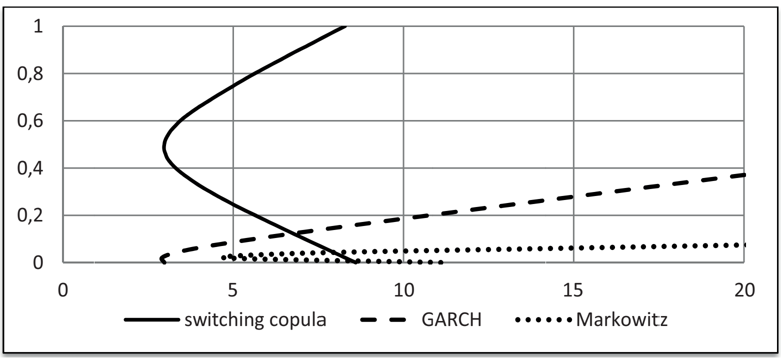

Figure 5. Efficient frontiers of Value at Risk for DAX/DJIA pair

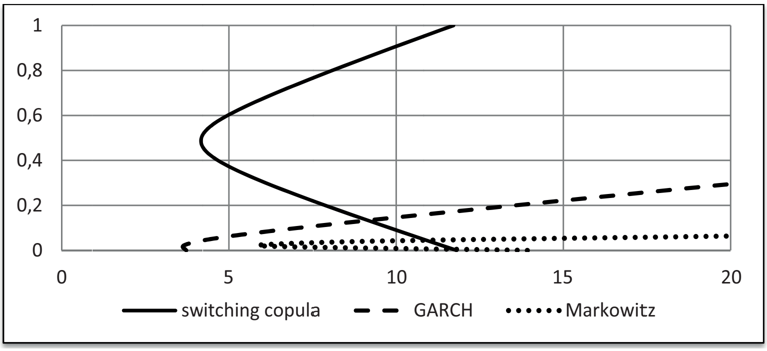

Figure 6. Efficient frontiers of Expected Shortfall for DAX/DJIA pair

Relatively small means of returns, presented in Table 2 cause a rapid increase of risk with increasing an expected portfolio return for the Markowitz model. Clearly, by definition, for every pair and every model ES is higher than VaR, see formula (33). Since negative expected returns are not interesting from a practical point of view, the included figures outline only the risks for positive expected returns.

For low expected returns (lower than 0.2 for ATX/DAX and ATX/DJIA pairs and lower than 0.05 for DAX/DJIA pair), the mean-variance model underestimates risks and after reaching some level overestimates them. The similar relation is observed for the GARCH model applied for the DAX/DJIA pair, but for the higher level. For ATX/DAX and ATX/DJIA pairs, the multivariate GARCH model underestimates risks for almost every level.

The level of an expected return, for which the minimum of a risk is attained, is determined by the forecast's multidimensional mean. At this particular time, means of all the three indices are the lowest for the Markowitz model, means of ATX/DAX and ATX/DJIA pairs are at similar levels for the switching copula model and the GARCH model.

With increasing of the expected return, VaR and ES increase with the similar speed for models based on a normal distribution. However, for all three pairs, ES increases essentially faster than VaR in the case of copula based model. A positive tail dependence in switching copula models and relatively fat tails of marginal distributions, such as a skewed *t* distribution, are reasons for this observation.

7. Conclusions

Recent contributions suggest non-normal distributions of multivariate asset's returns. Evidences for an asymmetry in univariate distributions and in dependences have been found. Furthermore, the kurtosis of an univariate distribution and extreme dependences are found to be greater than under the assumption of normal distribution. In the three analysed pairs of assets, all of these anomalies have been detected. Any model in which the conditional distribution is assumed to be normal does not fit since statistical tests reject hypothesis of normal distributions.

For the three pairs under study a switching copula models fit well. This model includes asymmetries and fat tails for both margins and for dependences. Conducted statistical tests confirmed goodness of fit for the switching copula models. Comparing results of a risk calculation, for the GARCH model and the Markowitz model to the switching copula model, we observed discrepancies.

A mean-variance model does not assume a dynamic structure of series, the expected mean of the series is significantly different for a dynamic model. Thus, a multivariate GARCH and a switching copula models forecast the mean at the similar level, while the estimated mean, using Markowitz model, stands out.

Misspecifications may cause both, an underestimation and an overestimation of a risk. Slopes of efficient frontiers describe the speed of increase of a risk with increasing expected return. It is observed that slopes for models which neglect anomalies, such as asymmetries and fat tails, are biased. In particular, a change of slope with the increasing expected return is underestimated.

Evaluations differ particularly for the Expected Shortfall risk. A tail's dependences and fat tails are ignored in models based on a normal distribution. Expected Shortfall measures not only a frequency of a loss, but also its size. The supposition that observed anomalies of the multivariate distribution of an assets' returns vector affects the size of an extreme return is confirmed.

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