Managerial Economics
2026, Vol. 27, No. 2, pp. 151–191
https://doi.org/10.7494/manage.2026.27.2.151
©2026 Authors. Creative Commons CC-BY 4.0

Jerzy Rydlewski[1], Przemysław Jaśko[2]

Goodwin model for France's business cycles. A functional data analysis approach

1. Introduction

In our work, we focus on the application of ODE (ordinary differential equations) systems with measurement error in economic research, exemplified by the empirical version of the Goodwin model of business cycles (Goodwin 1967), involving two state variables: labor's share of income, and employment ratio. We present an ODE system structural parameter estimation method from data with measurement noise, called the parameter cascade method, introduced by Ramsay and colleagues (Ramsay et al. 2007), which is a generalized profiling estimation procedure (Cao, Ramsay 2007), in which a subset of parameters are estimated as a function of the remaining parameters. This parameter cascade method of estimating the ODE parameters belongs to the group of collocation or functional basis expansion methods. The parameter cascade method takes a functionally oriented approach, in which approximation functions (defined by coefficients of basis functions defining a finite-dimensional functional space) are used to approximate the solutions of the ODEs. The estimation procedure involves two objective functions: inner, and outer. The inner objective function, from which the basis expansion coefficients (nuisance parameters) are obtained by minimization (conditionally on the current values of the ODE system structural parameters), involves a penalty that regularizes the approximations by controlling the extent to which the approximation functions fail to satisfy the ODEs. In turn, the outer objective function is minimized with respect to the ODE system structural parameters, and the fact that the basis expansion coefficients are implicit functions of the structural parameters is taken into account in the gradient-based iterative

[1] AGH University of Krakow, Krakow, Poland, e-mail: ry@agh.edu.pl, ORCID: 0000-0002-2071-0646
[2] Krakow University of Economics, Krakow, Poland, e-mail: jaskop@uek.krakow.pl, ORCID: 0000-0002-5534-9096

procedure. We briefly consider ODE system structural identifiability, which is the theoretical ability to uniquely recover model parameters from ideal, noise-free data, and is a prerequisite for reliable parameter estimation in empirical modeling. Structural parameters of the two-equation ODE system in the standard form of the Goodwin model are not identifiable (we present results on the global identifiability of the structural parameters, obtained with use of the procedure proposed by Dong and colleagues (Dong et al. 2023), called "differential elimination for dynamical models via projections with applications to structural identifiability"). So, to make it possible to estimate the structural parameters of the parameter cascade method, we propose to extend the Goodwin model to a three-equation ODE system, with one additional differential equation involved in the Goodwin model's derivation, representing labor productivity growth. Next, we briefly present a formal statement of the parameter cascade method for estimating ODE system structural parameters, comparing it to the NLS (nonlinear least squares estimator), which belongs to the estimation methods involving time discretization of ODEs.

The structure of the paper is as follows: in Section 2, we present a brief formal statement of the Goodwin model. In Section 3, we consider the structural parameter identifiability problem in the two-equation ODE system of the Goodwin model and propose a possible solution to it. Section 4 briefly describes the estimation procedure of the empirical version of the Goodwin model with the parameter cascade method, with comparison to the NLS estimation. Section 5 presents a simulation and empirical study of the Goodwin model estimation. For the simulation study, we present results on the finite-sample properties of the parameter cascade method and the NLS estimator for the three-equation ODE system of the extended Goodwin model. In the empirical part, we estimate the structural parameters of the three-equation ODE system Goodwin model with additive measurement noise using the parameter cascade method and NLS. It is based on the annual time series data for France over the period 1962–2009, representing three variables: labor productivity, labor's share of income, and employment ratio. The paper concludes with a conclusion section.

2. The Goodwin model

The Goodwin model (proposed in Goodwin 1967) is an economic growth cycle model consisting of two nonlinear ODE equations for two state variables: u and v, which are, respectively, the wage share in the national GDP, and the employment ratio. In the empirical version of the model, measurement noise is added to the two state variables, resulting in measurement equations.

The formal Goodwin model is a reparameterization of the Lotka–Volterra system of differential equations. The differential equations are rewritten such that they can be inferred from the differential and balancing equations, which are assumed on the grounds of economic theory.

The Goodwin model describes a single-sector, non-monetary economy in which there are only two groups: capitalists and workers. They produce a single good, which they can either consume or invest. The price of the good is normalized and equal to one. A description of the Goodwin model from the perspective of mathematical economics can be found, among others, in literature (Jakimowicz 2003).

Following Guzowska (Guzowska 2011), we present a sketch of the derivation of the Goodwin model as a two-equation ODE system from the assumed economic theory equations.

The following variables are involved in the derivation of the model: Y: output, K: capital stock, S: savings, L: employment, $a = Y/L$: labor productivity, n: labor supply, u: workers' share in the output (called labor's share of income), $v = L/n$: employment ratio, w: wage rate.

The main hypotheses defining the Goodwin model structure are described in the following points.

1. There are two homogeneous factors of production: labor and capital. Technological progress is neutral in the Harrod sense, with a constant coefficient:

$$\frac{\dot{a}}{a} = \alpha, \alpha > 0 \tag{1}$$

2. There is constant growth of the labor force:

$$\frac{\dot{n}}{n} = \beta, \beta > 0 \tag{2}$$

All wages are consumed, and all income is invested; all quantities are real and net:

$$S = I = \dot{K} = Y - wL \tag{3}$$

3. Real wages change in accordance with the Phillips curve, which represents the bargaining equation in the labor market:

$$\frac{\dot{w}}{w} = \rho v - \gamma, \rho, \gamma > 0 \tag{4}$$

4. A fraction of income is transferred to employees according to the formula:

$$u = \frac{wL}{Y} = \frac{w}{a} \tag{5}$$

After some straightforward transformations, we arrive at the final Goodwin model, which is the following two-equation ODE system:

$$\begin{cases} \dfrac{du}{dt} = u\left(\rho v - \gamma - \alpha\right) \\ \dfrac{dv}{dt} = v\left(-\dfrac{1}{\sigma}u + \dfrac{1}{\sigma} - \alpha - \beta\right) \end{cases} \tag{6}$$

where u is the wage share in the national GDP (called labor's share of income), and v is the employment ratio, i.e. the number of employed divided by the size of the labor force. All model parameters, i.e. α, β, γ, σ and ρ, should be positive and for the system to be economically meaningful, they also need to satisfy the following requirements: $\alpha + \beta < 1/\sigma$ and $\alpha + \gamma < \rho$. The productivity growth rate, denoted by α, is the rate of growth of labor productivity (e.g., the ratio of increase in output per worker per year). The parameter β describes the labor force growth rate. The parameter γ represents autonomous or institutional wage pressure, i.e. wage inflation expectations. The parameter ρ describes sensitivity of wage growth to employment, i.e. wage bargaining strength. The parameter σ is the capital-to-output ratio, which is also interpreted as the inverse of capital productivity.

To calculate the two-equation ODE system fixed points, we set:

$$\begin{cases} 0 = u\left(\rho v - \gamma - \alpha\right) \\ 0 = v\left(-\dfrac{1}{\sigma}u + \dfrac{1}{\sigma} - \alpha - \beta\right) \end{cases} \tag{7}$$

As solutions, we obtain two fixed points of the system of the state variables u and v:

- a hyperbolic fixed point at the origin (0, 0),
- a stable fixed point (u^*, v^*), which is the center of the closed orbits in the local vicinity

$$\begin{cases} u^* = 1 - \left(\alpha + \beta\right)\sigma \\ v^* = \dfrac{\alpha + \gamma}{\rho} \end{cases} \tag{8}$$

Orbits around this fixed point are closed and elliptical, thus the solutions are periodic, oscillating on an ellipse centered at the point (u^*, v^*).

To formalize these orbits, time is eliminated from the system's two differential equations to produce a single differential equation:

$$\frac{dv}{du} = \frac{v}{u}\frac{-\dfrac{1}{\sigma}u + \dfrac{1}{\sigma} - \alpha - \beta}{\rho v - \alpha - \gamma} \tag{9}$$

relating the variables u and v. The solutions of this equation are closed curves. It can be solved using the method of separation of variables, i.e. integrating:

$$\frac{\rho v-\alpha-\gamma}{v}dv+\frac{\frac{1}{\sigma}u-\frac{1}{\sigma}+\alpha+\beta}{u}du=0 \tag{10}$$

yields the implicit relationship:

$$C=\frac{1}{\sigma}u+\left(-\frac{1}{\sigma}+\alpha+\beta\right)\ln(u)+\rho v-(\alpha+\gamma)\ln(v) \tag{11}$$

where C is a constant value determined by the initial conditions and remains conserved along each curve.

The stability of the fixed points can be determined using ODE system linearization, which involves partial derivatives.

The Jacobian matrix of the two-equation system for state variables u and v is:

$$\mathbf{J}(u,v)=\begin{bmatrix}\rho v-\alpha-\gamma & \rho u\\ -\frac{v}{\sigma} & \frac{(1-u)}{\sigma}-(\alpha+\beta)\end{bmatrix} \tag{12}$$

When evaluated at the fixed point (0, 0), the Jacobian matrix **J** becomes:

$$\mathbf{J}(0,0)=\begin{bmatrix}-\alpha-\gamma & 0\\ 0 & \frac{1}{\sigma}-(\alpha+\beta)\end{bmatrix} \tag{13}$$

The eigenvalues of this matrix are: $\lambda_1=-\alpha-\gamma,\ \lambda_2=\frac{1}{\sigma}-(\alpha+\beta)$.

Taking into account the parameter restrictions assumed in the model, which imply that $\alpha+\gamma$ and $1/\sigma-(\alpha+\beta)$ are always greater than zero, the signs of the eigenvalues always differ. Hence the fixed point at the origin is a saddle point.

Evaluating the Jacobian **J** at the second fixed point, which is:

$$(u^*,v^*)=\left(1-(\alpha+\beta)\sigma,\frac{\alpha+\gamma}{\rho}\right) \tag{14}$$

results in:

$$\mathbf{J}(u^*,v^*)=\begin{bmatrix}0 & \rho-(\alpha+\beta)\sigma\rho\\ -\frac{\alpha+\gamma}{\sigma\rho} & 0\end{bmatrix} \tag{15}$$

The matrix considered here has the following conjugated complex eigenvalues:

$$\lambda_1=i\sqrt{(\alpha+\gamma)\left(\frac{1}{\sigma}-(\alpha+\beta)\right)},\ \lambda_2=-i\sqrt{(\alpha+\gamma)\left(\frac{1}{\sigma}-(\alpha+\beta)\right)}$$

As a result, orbits about the fixed point are closed and elliptic, the solutions are periodic, oscillating on an ellipse around the fixed point with frequency:

$$\omega = \sqrt{\lambda_1 \lambda_2} = \sqrt{(\alpha+\gamma)\left(\frac{1}{\sigma} - (\alpha+\beta)\right)} \text{ and period of } T_G = \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}}$$

Thus, the period T_G of these oscillations is given by:

$$T_G = \frac{2\pi}{\left[(\alpha+\gamma)\left(\frac{1}{\sigma} - (\alpha+\beta)\right)\right]^{1/2}} \tag{16}$$

As mentioned earlier, the two-equation ODE system of the Goodwin model can be seen as a reparameterization of the Lotka–Volterra model in its standard form.

The Lotka–Volterra model is defined as:

$$\begin{cases} \dfrac{du}{dt} = -(a_1 - b_1 v)u \\ \dfrac{dv}{dt} = (a_2 - b_2 u)v \end{cases} \tag{17}$$

This model is related to the Goodwin model by the following parameter relations:

$$a_1 = \alpha + \gamma, b_1 = \rho, a_2 = \frac{1}{\sigma} - (\alpha+\beta), b_2 = \frac{1}{\sigma} \tag{18}$$

3. Structural parameters identifiability problem in the Goodwin model two-equation ODE system

In this work, we consider linear and nonlinear dynamic state-space ODE models, depending on unknown parameters. Usually, these parameters are unknown and cannot be pre-specified and need to be estimated from data obtained by measuring the observable model variables (inputs and outputs). Output variables are given by the so-called measurement equations of the system, which are functions of the system's state variables and measurement noise (possibly additive). In order to solve the estimation problem, input–output data are collected.

In the case of the Goodwin model, the state variables of the ODE system consist of no input variables; only state variables. The measurement equation system involves the identity function, and an independent stationary Gaussian process is added to each equation.

A fundamental prerequisite for parameter estimation to be well-posed is the global identifiability of the parametric model. It should be noted that global identifiability assumes an ideal scenario with a perfect, error-free model structure and noise-free measurements. Therefore, while it is a necessary condition, it does not guarantee that accurate parameter identification from real-world input–output data will be achievable; factors such as measurement noise or poor problem conditioning can still impede successful identification (see Denis-Vidal and colleagues (Denis-Vidal et al. 2003). If there are at most a finite number of indistinguishable parameter values, the system is called locally (or nonuniquely) identifiable from the input–output data. Meanwhile, if there is an infinite number of different indistinguishable parameter values, the system is commonly called non-identifiable or unidentifiable.

Approaches used in the ODE structural parameter identifiability analysis include:

- Taylor series method,
- generating series method (GenSSI),
- similarity transformation approach,
- direct test method,
- differential algebra approach (implemented in DAISY, COMBOS, and Structural Identifiability Toolbox).

In this work, we make use of a differential algebra approach to assess the identifiability of the Goodwin model ODE system, by employing the method proposed by Dong and colleagues (Dong et al. 2023), referred to as differential elimination for dynamical models via projections with applications to structural identifiability. Differential algebra involves structures such as differential rings and fields, differential polynomials, rings of differential polynomials, differential ideals, and fields of definition, which are outside the scope of this work. The algebraic apparatus of differential algebra permits one to deal with polynomials which may involve derivatives of some variables (typically the state), and allows polynomials themselves to be differentiated, still keeping a formal similarity with classical commutative algebra. Formal treatments of differential algebra approaches in the context of the global or local identifiability of ODE systems can be found in literature (Hong et al. 2020; Dong et al. 2023; Sedoglavic 2002). A general treatment of differential algebra concepts can be found in Kolchin' work (Kolchin 1973).

The differential algebra approaches involve the idea of finding the input–output representation, which is to eliminate all state variables to derive equations that involve only model parameters, inputs, and observable outputs (and their derivatives). Ideally, a minimal set of such equations is derived. This approach also provides insights into how model parameters interact with each other. For

differential algebra approaches, structural identifiability reduces to a field membership problem: determining whether each parameter lies in the differential field generated by the observed outputs and their derivatives.

In the first phase of the procedure based on differential algebra, a set of generators is computed of the field of definition F of the ideal of the input–output relations; that is, the minimal subfield of the field of rational functions in parameters sufficient to write down the generators of this ideal. These generators are typically taken to be the coefficients of a finite set of input–output relations, which can be computed using:

- a characteristic set, as in DAISY (Differential Algebra for Identifiability of SYstems),
- a Gröbner basis (as in COMBOS),
- elimination by hand.

After this, in the second phase, the identifiability of a parameter is assessed by testing whether it belongs to F.

As already mentioned, to assess the structural identifiability of ODE systems, we use differential elimination for dynamical models via projections, as proposed by Dong and colleagues. (Dong et al., 2023). This algorithm computes a description of the set of differential-algebraic relations between the input and output variables of a dynamical system model. Based on this elimination procedure, a randomized algorithm is constructed to assess the structural identifiability of a parameter in a parametric model. The algorithm is implemented in the `Julia` package named `StructuralIdentifiability.jl`. A description of the usage of this package can be found in literature (Liyanage et al. 2025).

4. Estimation of the empirical version of the Goodwin Model with the parameter cascade method, with comparison to the NLS estimation

Procedures for estimating the parameters of an ODE from noisy data generally fall into three broad categories:

- linearization methods,
- discretization methods,
- collocation methods (basis function expansion methods).

In linearization methods, the ODE is initially approximated by a linear equation around a fixed point. Solutions derived from exponential basis functions are then obtained from this linearized form, serving as the foundation for estimating the model parameters.

The main representatives of estimation procedures involving discretization methods for obtaining approximate solutions to ODE systems are:

- NLS (Nonlinear Least Squares),
- MLE (Maximum Likelihood Estimation).

In discretization approaches, estimating the parameters of an ODE model involves iteratively solving the system numerically and updating the ODE parameters based on the resulting solution. This updating process aligns with the nonlinear regression framework. Specifically, the goal is to find the parameter vector $\boldsymbol{\theta}$ that minimizes the sum of the squared differences between the observed data and the solutions generated by the ODE model. The initial conditions – values of the state variables at the starting time – determine the unique solution obtained through numerical integration. However, in practical scenarios, these initial conditions are often unknown and need to be estimated prior to solving the ODEs. When the initial conditions are unavailable, they are incorporated into the parameter vector and estimated simultaneously with the ODE parameters.

When assumptions other than a stationary Gaussian process are made about the measurement error process, which is superimposed on the solution of the ODE system for state variables, the NLS can be replaced with the MLE estimation. The MLE procedure, when constructing the likelihood function, takes into account the specific distribution of variables in the measurement error process.

The NLS and MLE methods for ODE structural parameter estimation are implemented in the `fitode` package for `R` developed by Park (Park 2022). To obtain approximate time-discretized solutions of the considered ODEs, it makes use of ODE numerical solving procedures implemented in the `deSolve` package of `R` (authored by Soetaert and colleagues (Soetaert et al. 2010)).

In turn, collocation methods express the approximation of the ODE solution in terms of a basis function expansion.

In the initial phase of the development of collocation methods, a two-stage approach was proposed by Varah (Varah 1982), which employed nonparametric techniques to interpolate the derivatives of state variables over time. In the second stage, the parameters are estimated within a nonlinear regression framework, treating these interpolated derivatives as responses. Replacing numerical solutions with nonparametric estimates significantly reduces the computational demands; however, this substitution introduces estimation errors during the interpolation of derivatives, since direct observations of these derivatives are unavailable. Additionally, Cao and colleagues (Cao et al. 2012) demonstrated that such derivative estimation errors can cause bias in the estimation of ordinary differential equation (ODE) parameters. In contrast to the two-step method, Ramsay and colleagues

(Ramsay et al. 2007) proposed an alternative approach known as the parameter cascade method.

The parameter cascade method for ODE structural parameter estimation based on data with measurement noise was implemented in the following `R` packages: `pCODE` by Wang and Cao (Wang, Cao, 2019) (see also (Wang Cao 2023)) and `CollocInfer` by Hooker and colleagues (Hooker et al., 2016). As an alternative approach, Bayesian inference methods on the ODE system parameters are implemented in the `R` package `BayesianFitForecast` by Karami and colleagues (Karami et al. 2025).

Now we briefly describe the NLS and parameter cascade method of the ODE system structural parameter estimation. These methods are, respectively, representatives of the discretization and collocation methods.

4.1. NLS estimation of ODE structural parameters

In the NLS estimation procedure, first, numerical approximate solutions to the ODE initial value problem are obtained for given values of the structural parameters. From this approximate solution, values of the ODE state variables are taken at time points for which empirical time series data is available, and the sum of the squared errors between the approximate and observed values are calculated. The resulting sum of squared errors is an objective function, which is subject to minimization over ODE structural parameters (and optionally also over initial value parameters). Iterative procedures such as Levenberg–Marquardt can be used to minimize the objective function of the NLS estimator.

For an approximate numerical ODE solution, procedures based on linear multistep methods, such as LSODA introduced in literature (Petzold 1983), can be used.

Numerical methods for ODE approximate the solutions to initial value problems of the form:

$$\dot{\mathbf{x}} = \mathbf{f}\left(t, \mathbf{x}\right), \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \tag{19}$$

where $\mathbf{x}\left(t\right) = \left(x_1\left(t\right), \ldots, x_I\left(t\right)\right)'$, and $\mathbf{f}\left(t, \mathbf{x}\right) = \left(f_1\left(t, \mathbf{x}\right), \ldots, f_I\left(t, \mathbf{x}\right)\right)'$

The result is approximations for the value of $\mathbf{x}(t)$ at discrete times t_j: $\mathbf{x}(t) \approx \mathbf{x}(t_j)$ where $t_j = t_0 + jh$, where h is the time step and j is an integer.

Multistep methods use information from the previous s steps to determine the next value. Specifically, a linear multistep method combines linear combinations of past solution values x_j and their derivatives $\mathbf{f}(t_j, \mathbf{x}_j)$ to compute the current step's value.

The objective function for the ODE system can be written as:

$$S(\mathbf{y};\boldsymbol{\theta}) = \sum_{i=1}^{I}\sum_{j=0}^{n}\left(y_{ij} - x_i\left(t_{ij};\boldsymbol{\theta}\right)\right)^2 \tag{20}$$

where $x_i(t_{ij}; \boldsymbol{\theta})$ are the numerical approximate solutions of the ODE for the i-th state variable x_i, at the time moment t_{ij}, calculated under the structural parameter vectors $\boldsymbol{\theta}$, and y_{ij} is the j-th value of the i-th state variable, observed under additive measurement noise at time t_{ij}.

In an iterative procedure such as Levenberg–Marquardt, the gradient $\frac{\partial S}{\partial \boldsymbol{\theta}}$ of the objective function is used, which in turn, by the chain rule, involves the term $\frac{d\mathbf{x}}{d\boldsymbol{\theta}}$, which can be numerically approximated using LSODA numerical ODE solutions calculated under $\boldsymbol{\theta}_k$ (which are the parameter values from the k-th iteration), and under a slightly perturbed version.

Alternatively, gradients $\frac{d\mathbf{x}}{d\boldsymbol{\theta}}$ may be approximated by solving the sensitivity differential equations:

$$\frac{d}{dt}\left(\frac{d\mathbf{x}}{d\boldsymbol{\theta}}\right) = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{d\boldsymbol{\theta}},\ with \quad \frac{d\mathbf{x}}{d\boldsymbol{\theta}}|_{t=0} = \mathbf{0} \tag{21}$$

In the event that $\mathbf{x}(0) = \mathbf{x}_0$ must also be estimated, the corresponding sensitivity equations are:

$$\frac{d}{dt}\left(\frac{d\mathbf{x}}{d\mathbf{x}_0}\right) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{d\mathbf{x}_0},\ with \quad \frac{d\mathbf{x}}{d\mathbf{x}_0}|_{t=0} = \mathbf{I} \tag{22}$$

4.2. Parameter cascade method estimation of ODE structural parameters

The parameter cascade method proposed by Ramsay and colleagues (Ramsay et al. 2007) belongs to the family of collocation methods that express the approximation $\hat{x}_i$ of x_i in terms of a basis function expansion. The parameter cascade method provides an estimation of ODE parameters from the noisy observations.

In the parameter cascade method, there is a member of basis function expansion or collocation methods. From such methods, the finite-dimensional function space L_p on [0,T] is first chosen. Approximation functions to the solutions of the ODEs will be chosen from L_p (this is equivalent to choosing the coefficients of the basis functions). For example, using a spline basis to approximate an initial value problem ODE solution is equivalent to the use of an implicit Runge-Kutta method for stepping points located at the knots defining the basis (see (Deuflhard, Bornemann 2012).

A general overview of functional data analysis (FDA) can be found in literature (Ramsay, Silverman 2005; Rydlewski 2025) also presents FDA applications in economic research. The topic of functional data smoothing is treated there in detail. A software implementation of FDA is covered, among others, in work of Ramsay and colleagues (Ramsey et al. 2009). The FDA procedures considered there are implemented in the `R` package `fda` and related family of packages.

In the case of the parameter cascade method, two groups of parameters can be distinguished: structural parameters $\boldsymbol{\theta}$ of the ODE system, and basis function expansion coefficients of the smoothed approximations $\hat{x}_i$ (for each i-th state variable of the ODE system), embraced in the vector $\mathbf{c}$, which are called the nuisance parameters since they are not essential for estimating the ODE model but rather interpolate the observations. For each given set of ODE parameters $\boldsymbol{\theta}$, the basis coefficients $\mathbf{c}$ are estimated by minimizing the inner objective function $J(\boldsymbol{c} \mid \boldsymbol{\theta})$. Thus, the estimate for the basis coefficients becomes an implicit function of the ODE parameters $\boldsymbol{\theta}$ as $\mathbf{c}(\boldsymbol{\theta})$. The amount of regularization is controlled by smoothing parameters in vector $\boldsymbol{\lambda}$. Structural parameters are estimated by minimizing the outer objective function $H(\boldsymbol{\theta}; \boldsymbol{\lambda})$ over $\boldsymbol{\theta}$.

Now we present the main aspects of the parameter cascade method, following Ramsay and colleagues. (Ramsey et al. 2007) and Wang and Cao (Wang, Cao 2023).

We consider the ODE system in the following form:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, u, t \mid \boldsymbol{\theta}) \tag{23}$$

where $\mathbf{x}(t)$ denotes the fully or partially observed states of the system at time t.

The considered ODE system links the first derivative over time of the states with $\mathbf{x}$ itself and other influencing components u through some function f defined by the parameter vector $\boldsymbol{\theta}$.

Each of $x_i(t)$ is expressed as a linear combination of basis functions:

$$x_i(t) = \sum_{k}^{K_i} c_{ik}\phi_{ik}(t) = \mathbf{c}_i'\boldsymbol{\phi}_i(t) \tag{24}$$

where K_i is the number of basis functions, and $\boldsymbol{\phi}_i(t)$ is the vector of basis functions evaluated at a time point t.

We define the joint vector of all involved basis coefficients as $\mathbf{c} = \left(\boldsymbol{c}_1', \mathbf{c}_2', \ldots, \mathbf{c}_I'\right)'$, and accordingly, the joint vector of all involved basis function values at time t as $\boldsymbol{\phi}(t) = \left(\boldsymbol{\phi}_1'(t), \boldsymbol{\phi}_2'(t) \ldots, \boldsymbol{\phi}_I'(t)\right)'$.

Let $y_{ij} = x_i(t_{ij}) + e_{ij}$ be the j-th noisy observation of the i-th state variable at time t_{ij}, where e_{ij} is the measurement error or noise, $j = (1, \ldots, n_i)$. It is assumed that vector $\mathbf{e}_i = (e_{i1}, \ldots, e_{in_i})$ of random errors has a distribution function $g(\mathbf{e}_i)$ with

the distribution parameter σ_i. The ODE system structural parameters need to be estimated from noisy observations of the system.

The inner objective function is defined as:

$$J(\mathbf{c}\,|\,\boldsymbol{\theta}) = \sum_{i=1}^{I}\left[-\ln g(\mathbf{e}_i) + \lambda_i \int \left(\dot{x}_i(t) - f_i(\mathbf{x}(t), t\,|\,\boldsymbol{\theta})\right)^2 dt\right] \tag{25}$$

where $\int \left(\dot{x}_i(t) - f_i(\mathbf{x}(t), t\,|\,\boldsymbol{\theta})\right)^2 dt$ is the smoothness penalty defined by the discrepancy between the time derivatives of the smoothed functions and the ODE model equations, for each state variable x_i. Each penalty is multiplied with a respective smoothing parameter λ_i, and they are combined over all dimensions.

For each given set of ODE parameters $\boldsymbol{\theta}$, the basis coefficients $\mathbf{c}$ is estimated by minimizing $J(\boldsymbol{c}\,|\,\boldsymbol{\theta})$ over $\mathbf{c}$. The estimate for the nuisance parameters $\mathbf{c}$ becomes an implicit function $\mathbf{c}(\boldsymbol{\theta})$ of the structural parameters $\boldsymbol{\theta}$. The inner objective function needs to be optimized whenever $\boldsymbol{\theta}$ is updated.

The upper-stream estimation of the parameter cascade method is focused on the structural parameter $\boldsymbol{\theta}$. To carry out this estimation, the outer objective function $H(\boldsymbol{\theta})$ is optimized with respect only to the structural parameters $\boldsymbol{\theta}$. By default, the outer objective function is defined as the negative log-likelihood function or the sum of the squared errors, given the distributions of random errors, namely:

$$H(\boldsymbol{\theta}) = -\sum_{i=1}^{I} \ln g(\hat{\mathbf{e}}_i\,|\,\boldsymbol{\theta}) \tag{26}$$

where $\hat{\mathbf{e}}_i = \boldsymbol{y}_i - \hat{x}_i(\mathbf{t}_i\,|\,\boldsymbol{\theta})$. Here, $\hat{x}_i(\mathbf{t}_i\,|\,\boldsymbol{\theta}) = \hat{\mathbf{c}}_i'(\boldsymbol{\theta})\boldsymbol{\phi}_i(\mathbf{t}_i)$, where $\hat{\mathbf{c}}_i'(\boldsymbol{\theta})$ is the optimizer of $J(\boldsymbol{c}\,|\,\boldsymbol{\theta})$, given the current $\boldsymbol{\theta}$.

In the considered profiling procedure, the nuisance parameter estimates, as already mentioned, are defined to be implicit functions $\hat{\mathbf{c}}_i(\boldsymbol{\theta}, \boldsymbol{\sigma}; \boldsymbol{\lambda})$ of the structural parameters, such that, each time $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ are changed, an inner fitting objective $J(\hat{\boldsymbol{c}}\,|\,\boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ is reoptimized with respect to $\hat{\mathbf{c}}$ alone. The estimating function $\hat{\mathbf{c}}_i(\boldsymbol{\theta}, \boldsymbol{\sigma}; \boldsymbol{\lambda})$ is regularized by incorporating a penalty term in J that controls the size of the extent that $\hat{\boldsymbol{x}} = \hat{\mathbf{c}}'\boldsymbol{\phi}$ fails to satisfy the differential equation exactly, in a manner that is specified below. The amount of regularization is controlled by smoothing parameters in vector $\boldsymbol{\lambda}$. This is a process of eliminating the direct effect of the nuisance parameters on the fit of the model.

Finally, the data fitting outer criterion $H(\boldsymbol{\theta}, \boldsymbol{\sigma}; \boldsymbol{\lambda})$ is optimized solely with respect to the structural parameters. The outer objective H's dependence on $(\boldsymbol{\theta}, \boldsymbol{\sigma})$ is twofold: direct, and indirect through the involvement of the regularized estimates $\hat{\mathbf{c}}_i(\boldsymbol{\theta}, \boldsymbol{\sigma}; \boldsymbol{\lambda})$, which define the fitted values $\hat{x}_i$. Since $\hat{\mathbf{c}}_i(\boldsymbol{\theta}, \boldsymbol{\sigma}; \boldsymbol{\lambda})$ are already regularized, the criterion H does not need additional regularization and serves as

a straightforward measure of fit, such as the sum of squared errors, log-likelihood, or another appropriate metric based on the error distribution e_{ij}.

Hyperparameter $\boldsymbol{\lambda}$ may be adjusted manually by using numerical and visual heuristics. Alternatively, $\boldsymbol{\lambda}$ may be automatically estimated through the use of a measure $F(\boldsymbol{\lambda})$ of model complexity or mean-squared error, such as the generalized cross-validation criterion often used in least squares spline smoothing. In such a situation, the vector $\boldsymbol{\lambda}$ defines a third level of parameters and leads to the extended parameter cascade, in which structural parameter estimates are in turn defined to be functions $\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})$ and $\hat{\boldsymbol{\sigma}}(\boldsymbol{\lambda})$ of regularization or complexity parameters, and nuisance parameters now also become functions of $\boldsymbol{\lambda}$ by their dependence on structural parameters.

We now define the form of the gradient $\frac{dH}{d\boldsymbol{\theta}}$, which is used in the optimization of H with respect to $\boldsymbol{\theta}$. To simplify the notation, the dependence of objective H on $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ is dropped, and parameter $\boldsymbol{\lambda}$ is treated as fixed. The following results can be readily generalized to incorporate the joint estimation of the system parameters $\boldsymbol{\theta}$ and the error distribution parameters $\boldsymbol{\sigma}$. We assume that H is twice continuously differentiable with respect to both $\boldsymbol{\theta}$ and $\mathbf{c}$, and that the second partial derivative or Hessian matrices $\partial^2 H/\partial\boldsymbol{\theta}^2$ and $\partial^2 H/\partial\hat{\mathbf{c}}^2$ are positive definite over a non-empty neighborhood $\mathcal{N}$ of y in data space.

The gradient or total derivative with respect to $\boldsymbol{\theta}$ is:

$$\frac{dH}{d\boldsymbol{\theta}} = \frac{\partial H}{\partial\boldsymbol{\theta}} + \frac{\partial H}{\partial\hat{\mathbf{c}}}\frac{d\hat{\mathbf{c}}}{d\boldsymbol{\theta}} \tag{27}$$

Since $\hat{\mathbf{c}}(\boldsymbol{\theta})$ is not available explicitly, the implicit function theorem can be used to obtain:

$$\frac{d\hat{\mathbf{c}}}{d\boldsymbol{\theta}} = \left(\frac{\partial^2 J}{\partial\hat{\mathbf{c}}^2}\right)^{-1}\frac{\partial^2 J}{\partial\hat{\mathbf{c}}\partial\boldsymbol{\theta}} \tag{28}$$

$$\frac{dH}{d\boldsymbol{\theta}} = \frac{\partial H}{\partial\boldsymbol{\theta}} - \frac{\partial H}{\partial\hat{\mathbf{c}}}\left(\frac{\partial^2 J}{\partial\hat{\mathbf{c}}^2}\right)^{-1}\frac{\partial^2 J}{\partial\hat{\mathbf{c}}\partial\boldsymbol{\theta}} \tag{29}$$

Regarding the computational aspects, the optimization of the inner objective function follows the non-linear least squares (NLS) framework, which allows the use of the Levenberg–Marquardt algorithm to estimate $\hat{\mathbf{c}}_i(\boldsymbol{\theta})$. While the optimization of the outer objective function also resembles an NLS problem, the incorporation of a parameter cascade strategy introduces an additional layer of optimization at each outer estimation step, resulting in a nested optimization structure. This allows the use of the Levenberg–Marquardt method, but the Jacobian matrix needs to be calculated for each iteration of the update on the structural parameters, and each entry of the Jacobian for the outer objective involves a complete optimization of the inner one.

The variances of the parameter cascade method estimators of the ODE system structural parameters can be assessed using two methods: the bootstrap variance estimator and the Delta variance estimator, for which algorithms are described in detail, respectively, in literature (Ramsey et al. 2007; Wang, Cao 2023). Parameter cascade method estimation of the ODE structural parameters and nuisance parameters, with the two mentioned methods for variance estimations, among other procedures, is employed in the `R` package `pCODE` by Wang and Cao (Wang, Cao 2019). Procedures for optimally selecting the hyperparameter $\boldsymbol{\lambda}$ are implemented in the `R` package `CollocInfer` by Hooker and colleagues (Hooker et al. 2016).

The asymptotic efficiency and finite-sample properties of the parameter cascade method estimators of the ODE system parameters can be found in literature (Qi, Zhao 2010). They present formally derived asymptotic properties of the considered estimator, and the results for a finite sample based on simulations.

An empirical procedure for parameter cascade estimation of parameters of the predator–prey ODE system is shown in the paper by Cao and colleagues (Cao et al. 2008). Zhang and colleagues (Zhang et al. 2015) propose model selection and estimation in situations when there are competing ordinary differential equation (ODE) models (which are special cases of a "full" model). The consistent selection procedure employs statistical estimation of the full model, followed by a combination of a least squares approximation (LSA) and the adaptive Lasso. Selected model parameters are then estimated using the parameter cascade method.

5. Simulation and empirical study of the Goodwin model estimation

In the following section, we present both a simulation study and an empirical application of the Goodwin model. The finite-sample performance of the parameter cascade and nonlinear least squares (NLS) estimators for the three-equation Goodwin ODE system is first evaluated through simulations. Next, the proposed methods are applied to estimate the Goodwin model for the French economy over the period 1962–2009.

5.1. Simulation study comparing the parameter cascade method and NLS estimators for the three-equation ODE Goodwin model

We consider the Goodwin model represented by the four-state ODE system with additive *iiN* measurement noise with a standard deviation of 0.05. The four state variables of the ODE system are: (a, n, u, and v), which can be associated

with the following economic phenomena: a: labor productivity, n: labor force, u: labor income share, and v: employment rate. The assumed values of the structural parameters of the considered ODE system are as follows: $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, initial values: $a_0 = 1$, $n_0 = 1$, $u_0 = 0.4$, $v_0 = 0.4$.

The orbit cycle period, calculated using the assumed parameter values of the considered ODE system, is $T_G = 28.0993$.

The (u, v) subsystem has two fixed points: (0,0) and (0.5,0.5). We simulate $N_{rep} = 100$ replications from the four-equation ODE system of the Goodwin model, with additive measurement noise. For ODE parameters, $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$ are assumed, and additive measurement noise is specified for each of the four equations in the form of the following processes: $iiN(0, \sigma_j = 0.05)$, $j = a, n, u, v$. The following initial values for the trajectories are assumed: $u_0 = 0.4$, $v_0 = 0.4$. The length of the simulated trajectories is $T = 100$. In the following figures, for the 100 replicated simulations with additive measurement noise, we present the time series of a, n, u, v (Fig. 1), and (u, v) phase-space portraits (Fig. 2).

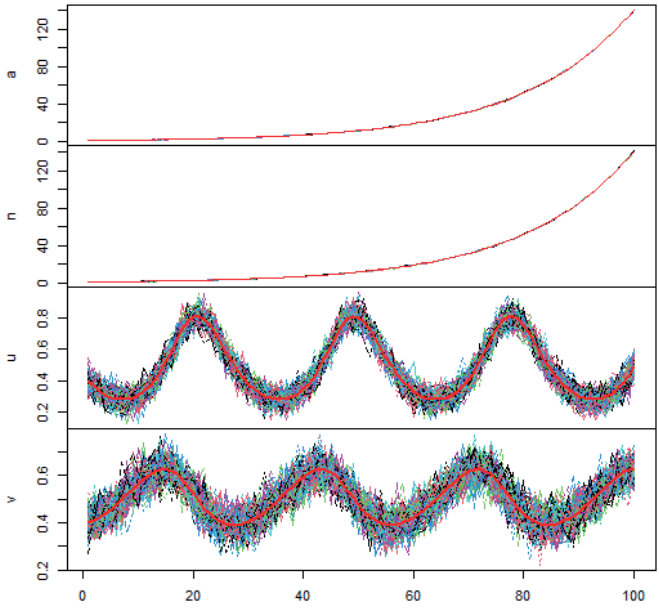

Figure 1. 100 simulated trajectories (of length $T = 100$) from the Goodwin model represented by the four-state equation ODE system with additive $iiN(0, 0.05)$ measurement noise; ODE system parameters: $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$; initial values: $a_0 = 1$, $n_0 = 1$, $u_0 = 0.4$, $v_0 = 0.4$

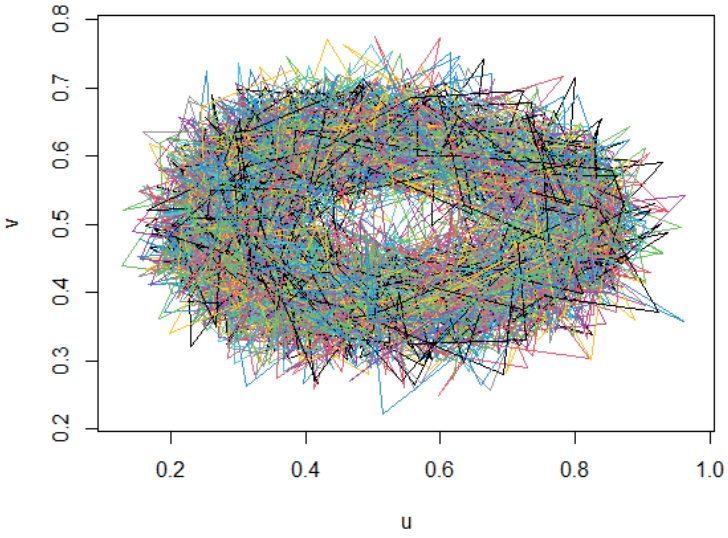

Figure 2. Phase plot in (u, v) space of the 100 simulations from the Goodwin model represented by the four-state equation ODE system with additive $iiN(0, 0.05)$ measurement noise; ODE system parameters: $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$; initial values: $u_0 = 0.4$, $v_0 = 0.4$

5.2. Estimation settings for structural parameters of the Goodwin ODE system based on simulated trajectory data

From each of the 100 simulated replications, the values of the following estimators are calculated: the simplified estimator, the NLS (nonlinear least squares) estimator, and the parameter cascade estimator. For the iterative procedures used to obtain the NLS and cascade parameter method estimators, the simplified estimator is used as the starting value.

The NLS and the parameter cascade method estimators are based on the data from the three-equation subsystem of state variables (a, u, v), with additive measurement noise $iiN(0, \sigma_j)$, $j = a, u, v$. Therefore to compute its values, we used 100 simulated trajectories of length $T = 100$ for (a, u, v). For the parameter cascade method estimator, we experimentally arrived at the following settings: B-spline bases for the smoothed functional representations of the trajectories for each of the three state variables a, u and v. For each of the three trajectories a, u and v, the assumed B-spline basis functions are of order four and have four knots, which implies there are 6 B-splines basis functions (number of B-splines basis functions is equal to: number of knots + order of basis functions – 2) in each basis.

Moreover, estimates for the cycle period T_G and for the coordinates of the fixed point (u^*, v^*) of the (u, v) subsystem, using the obtained values for the considered estimators, are provided.

In the following subparagraphs, we provide graphical presentations of the distributions of the mentioned estimates based on the simulation data from 100 replications.

5.3. Simplified estimator distribution from the simulation

The simplified estimates are generally not considered estimators *per se*, but more a starting point for iterative procedures of NLS and parameter cascade method estimation, based on the four-equation differential system for the state variables a, n, u, v:

$$\begin{cases} \dfrac{da}{dt} = \alpha a \\ \dfrac{dn}{dt} = \beta n \\ \dfrac{du}{dt} = u\left(\rho v - \gamma - \alpha\right) \\ \dfrac{dv}{dt} = v\left(-\dfrac{1}{\sigma}u + \dfrac{1}{\sigma} - \alpha - \beta\right) \end{cases} \tag{30}$$

Simplified estimate for the α parameter is:

$$\tilde{\alpha} := \frac{1}{T-1}\sum_{t=2}^{T}\frac{a_t - a_{t-1}}{a_{t-1}} \tag{31}$$

Simplified estimate for the β parameter is:

$$\tilde{\beta} := \frac{1}{T-1}\sum_{t=2}^{T}\frac{n_t - n_{t-1}}{n_{t-1}} \tag{32}$$

To obtain simplified estimates for the other parameters, we use the derivative equations for $u(t)$ and $v(t)$.

In the case of the derivative equation for $u(t)$:

$$\frac{du}{dt} = u\left(\rho v - \gamma - \alpha\right) \tag{33}$$

we rearrange it into the form:

$$\rho v - \gamma = \frac{1}{u}\cdot\frac{du}{dt} + \alpha \tag{34}$$

In the case of the derivative equation for $v(t)$:

$$\frac{dv}{dt} = v\left(-\frac{1}{\sigma}u + \frac{1}{\sigma} - \alpha - \beta\right) \tag{35}$$

we rearrange it into the form:

$$-\frac{1}{\sigma}u + \frac{1}{\sigma} - \alpha - \beta = \frac{1}{v}\cdot\frac{dv}{dt} \tag{36}$$

To obtain a simplified estimate of $\frac{1}{\sigma}$, we perform the following rearrangement:

$$\frac{1}{v}\cdot\frac{dv}{dt} + \alpha + \beta = \frac{1}{\sigma}\cdot(1-u) \tag{37}$$

The dependent variable for the linear regression without a constant, used for the initial guess of $\widetilde{\frac{1}{\sigma}}$, has the form:

$$\frac{v_t - v_{t-1}}{v_t} + \tilde{\alpha} + \tilde{\beta} \tag{38}$$

The independent variable for the linear regression without a constant, used for the simplified estimate of $\widetilde{\frac{1}{\sigma}}$, has the form: $1 - u_t$.

Simplified estimate for the σ parameter is:

$$\tilde{\sigma} := \frac{1}{\widetilde{\frac{1}{\sigma}}} \tag{39}$$

To provide the simplified estimates of ρ and $-\gamma$, we use the already established relation:

$$\frac{1}{u}\cdot\frac{du}{dt} + \alpha = \rho v - \gamma \tag{40}$$

The dependent variable for the linear regression used for the initial guess of $\tilde{\rho}$ and $\widetilde{-\gamma}$ has the form:

$$\frac{u_t - u_{t-1}}{u_t} + \tilde{\alpha} \tag{41}$$

The independent variable for the linear regression used for the initial guess of $\tilde{\rho}$ and $\widetilde{-\gamma}$ is: v_t.

OLS estimates of the linear regression considered here are: $\tilde{\rho}$ (intercept) and $\widetilde{-\gamma}$ (slope).

Thus, we take:

$$\tilde{\gamma} := \widetilde{--\gamma} \tag{42}$$

A high degree of caution is needed when proposing simplified estimates based on a time discretization of the Goodwin model system equations because, for example, the system resulting from the simple Euler time discretization of the Lotka–Volterra system can have different dynamic properties than the original continuous-time one (see Guzowska 2011).

In the following figures, we present the distribution of the simplified estimator values obtained from 100 simulated trajectories from the four-equation ODE system (for the state variables (a, n, u, v) with additive measurement noise. Figure 3 presents the boxplot of the simplified estimator values obtained from the simulated time series.

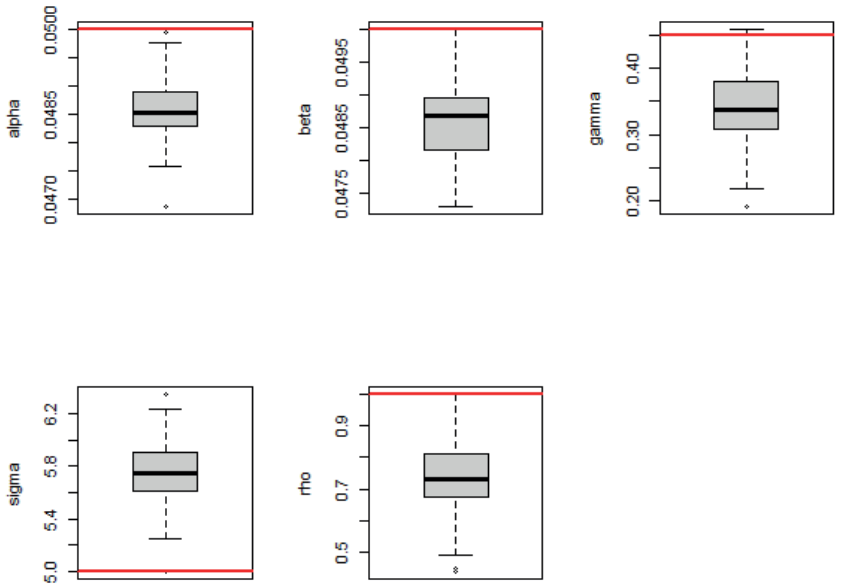

Figure 3. Boxplot of the simplified estimator values for the parameters α, β, γ, σ, ρ, for the ODE system of four equations for the state variables a, n, u, v, obtained from 100 simulated replications, with assumed parameter values: $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$; initial values: $u_0 = 0.4$, $v_0 = 0.4$ with trajectory length $T = 100$. True values of the parameters are represented by horizontal red lines

5.4. NLS (nonlinear least squares) estimator distribution from the simulation

In the NLS estimation procedure of the three-equation ODE system of the Goodwin model structural parameters, we combine the LSODA numerical ODE approximate solution procedure, implemented in the function `ode` from the `R` package `deSolve`, with the Levenberg–Marquardt algorithm (implemented by the function `nls.lm` from the `R` package `minpack.lm` by Elzhov et al. (2016)) for iteratively minimizing the nonlinear residual sum of squares objective function.

In the following figures, we present the distribution of the NLS estimator values obtained from 100 simulated trajectories from the three-equation ODE system (for the state variables (a, u, v) with additive measurement noise. Figure 4 presents the phase portrait for the (u, v) subsystem with parameters equal to the NLS estimator values obtained from the simulated trajectories. Figure 5 presents the boxplot of the NLS estimator values obtained from the simulated time series. Figure 6 presents the scatterplot of the fixed points of the (u, v) subsystem calculated using NLS estimator values.

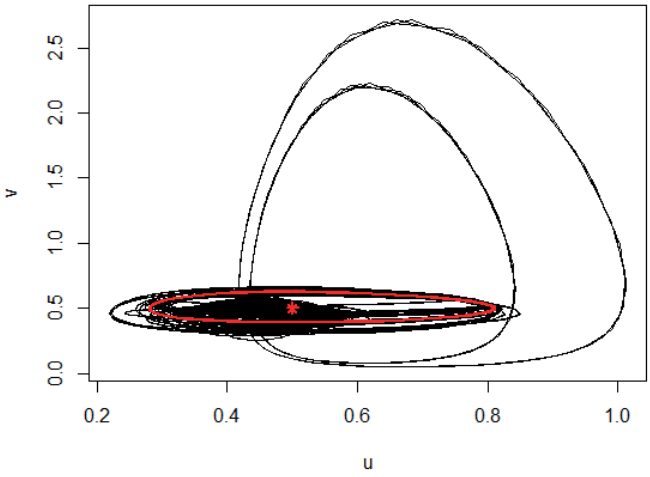

Figure 4. Phase plots for (u, v) generated from the two-equation ODE subsystem for state variables u and v, with parameter values obtained using NLS estimator from each of the 100 simulated replications, trajectories start from the observed values of u and v for each replication. Red bolded curves represent trajectories from the noiseless ODE system with the assumed parameters of $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, and the initial values $u_0 = 0.4$ and $v_0 = 0.4$ assumed in the simulation study. The red asterisk represents the true fixed point (0.5, 0.5), which is the center of the red orbit

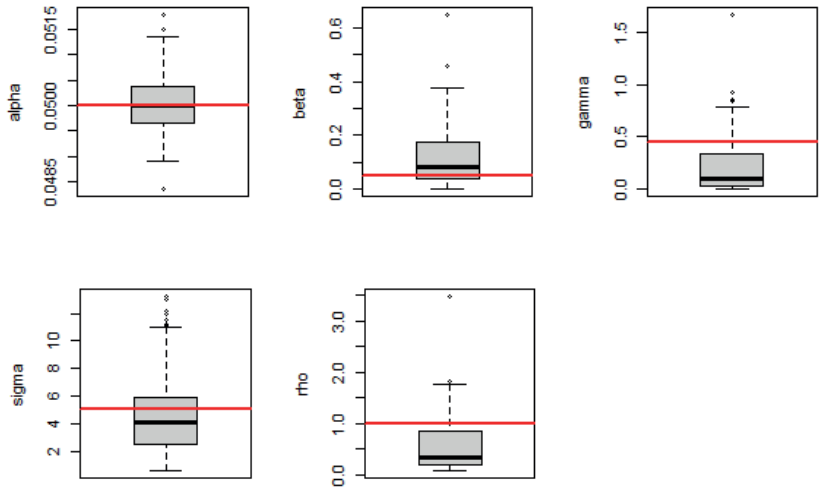

Figure 5. Boxplot of the NLS estimator values of the parameters α, β, γ, σ, ρ, for the three-equation ODE system for the state variables a, u, v, obtained from 100 simulated replications, with assumed parameter values: $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$; initial values $u_0 = 0.4$, $v_0 = 0.4$; with trajectory length $T = 100$. True values of the parameters are represented by horizontal red lines

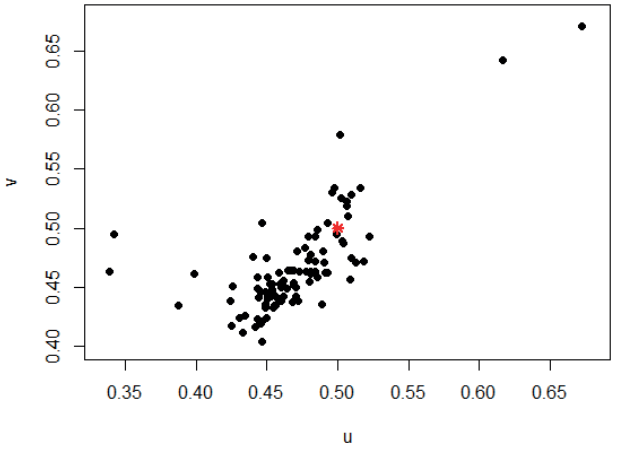

Figure 6. Scatterplot of the (u, v) subsystem fixed points calculated using NLS estimator values of parameters of the considered three-equation ODE system, obtained from 100 simulated replications. The true coordinates of the non-zero fixed point (0.5, 0.5) are represented by the red asterisk

5.5. Parameter cascade estimator distribution from the simulation

We take the following settings for the parameter cascade method estimation for the structural parameters of the three-equation ODE Goodwin system model for the state variables a, u and v, based on each of the 100 simulated trajectories starting from $(a_0, u_0, v_0) = (1, 0.4, 0.4)$ with additive $iiN(0, 0.05)$ measurement noise:

- times: $t \in [0, \ldots, 100]$,
- basis functions: B-splines,
- basis function order: 4,
- number of knots: 4,
- number of basis functions: number of knots + basis function order – 2 = 6,
- penalty hyperparameter λ (assumed jointly for all state variables) from the inner objective function: 10^{-2} (selected based on an experiment with λ from the grid: 10^k, $k \in [-5, \ldots, 5]$).

The following two plots present trajectories simulated from the (u, v) subsystem of the Goodwin model starting, respectively, from the observed initial values of u and v, and from the smoothed basis-expansion values of these initial conditions. To obtain each of the 100 trajectories, we use parameter cascade estimates based on a particular replication of simulated time series data (see Figures 7 and 8).

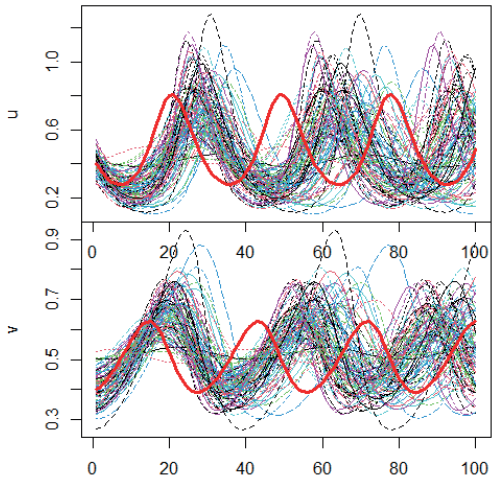

Figure 7. Trajectories (of length $T = 100$) generated from the two-equation ODE subsystem for state variables u and v, with parameter values estimated using the parameter cascade method, from each of the 100 simulated replications. Trajectories start from the observed values of u and v for the each replication. Red bold curves represent trajectories from the noiseless ODE system with the assumed parameters $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, and initial values of $u_0 = 0.4$, $v_0 = 0.4$

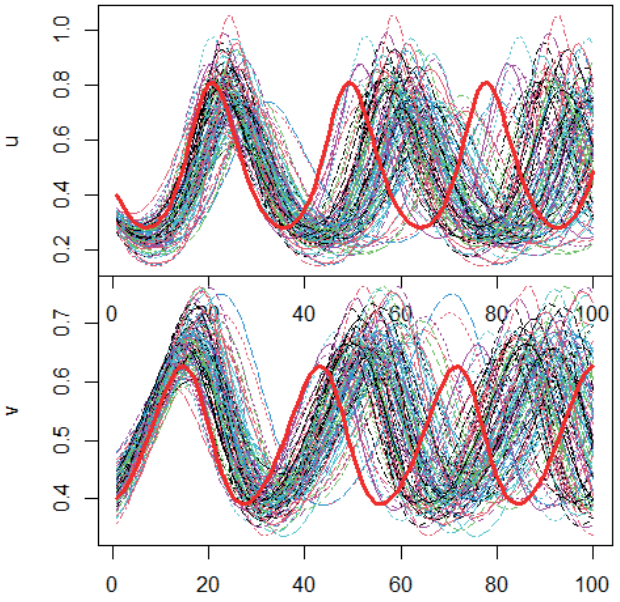

Figure 8. Trajectories (of length $T = 100$) generated from the two-equation ODE subsystem for state variables u and v, with parameter values estimated using the parameter cascade method, from each of the 100 simulated replications. Trajectories start from the smoothed values of u and v for the each replication. Red bold curves represent trajectories from the noiseless ODE system with the assumed parameters $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, and initial values of $u_0 = 0.4$, $v_0 = 0.4$

The following two plots show phase portraits simulated from the (u, v) subsystem of the Goodwin model starting, respectively, from the observed initial values of u and v and from smoothed basis-expansion values of these initial values (see Figures 9 and 10). To obtain each of the 100 trajectories, we use parameter cascade estimates based on a particular replication of simulated time series data.

In both figures, red curve represents the true orbit obtained from the noiseless ODE system with parameters of $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, and initial values of $u_0 = 0.4$, $v_0 = 0.4$, assumed in the simulation study. Red asterisk represents the true fixed point (0.5, 0.5), which is the center of the red line orbit.

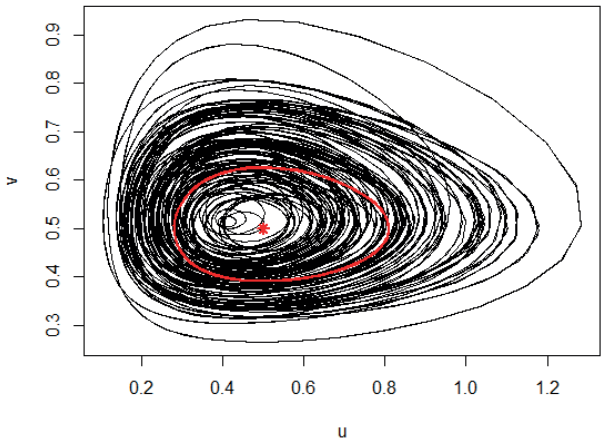

Figure 9. Phase plots for (u, v) generated from the two-equation ODE subsystem for state variables u and v, with parameter values estimated using the parameter cascade method, from each of the 100 simulated replications. Trajectories start from the observed values of u and v for the replication. The red curve represents the true orbit obtained from the noiseless ODE system with parameters of $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, and initial values of $u_0 = 0.4$, $v_0 = 0.4$, as assumed in the simulation study. The red asterisk represents the true fixed point (0.5, 0.5), which is the center of the red orbit

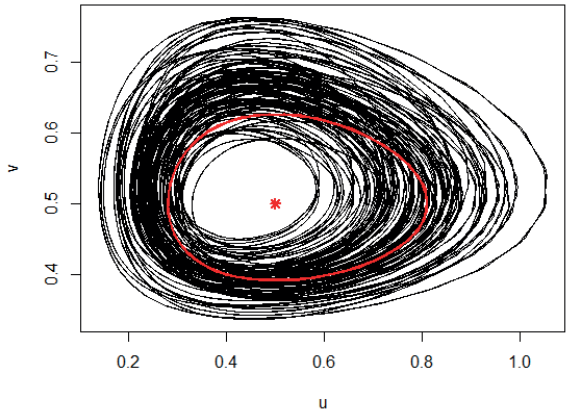

Figure 10. Phase plots for (u, v) generated from the two-equation ODE subsystem for state variables u and v, with parameter values estimated using the parameter cascade method, from each of the 100 simulated replications. Trajectories start from the smoothed values of u and v for each replication. The red curve represents the true orbit obtained from the noiseless ODE system with parameters of $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$, and initial values of $u_0 = 0.4$, $v_0 = 0.4$, as assumed in the simulation study. The red asterisk represents the true fixed point (0.5, 0.5), which is the center of the red orbit

Next, we present a boxplot of the parameter cascade estimator values obtained from the simulated time series (see Figure 11).

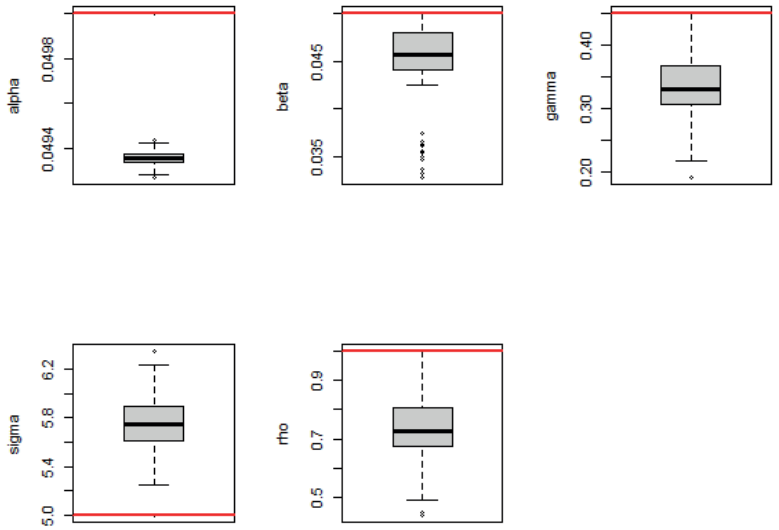

Figure 11. Boxplot of the parameter cascade method estimator values for the parameters α, β, γ, σ, ρ, for the three-equation ODE system for the state variables a, u, v obtained from 100 simulated replications, with assumed parameter values $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$; initial values $u_0 = 0.4$, $v_0 = 0.4$; with trajectory length $T = 100$. True values of the parameters are represented by horizontal red lines

5.6. Descriptive statistics for the values of the considered estimators obtained in a 100-replication simulation study

Using each of the 100 simulated replications of trajectories of length $T = 100$, from the (a, n, u, v) ODE system with additive *iiN* measurement noise, we computed values of the considered structural parameter estimators, i.e. the simplified estimator, the NLS estimator, and the parameter cascade method estimator (the latter two estimators are based on the three-variables subsystem a, u, v). Descriptive statistics are calculated for these 100 values of the considered structural parameter estimators for the Goodwin ODE model system. They are provided in the following tables, along with descriptive statistics for T_G and the fixed point (u^*, v^*) estimates, based on the three considered structural parameter estimators (see Tables 1–3).

Table 1

Estimates of the three-equation ODE system for the state variables a, u, and v of the Goodwin model, obtained using the simplified estimator, the NLS estimator, and the parameter cascade method estimator. The true values of the considered ODE system parameters are: $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.45$, $\sigma = 5$, $\rho = 1$

Parameter	n	Mean	Standard deviation	Median	Trimmed	MAD	Min	Max	Range	Skewness	Kurtosis	Standard error
Simplified parameter estimator												
α	100	0.05	0.00	0.05	0.05	0.00	0.05	0.05	0.00	–0.03	0.35	0.00
β	100	0.05	0.00	0.05	0.05	0.00	0.05	0.05	0.00	–0.09	–0.49	0.00
γ	100	0.34	0.06	0.34	0.34	0.05	0.19	0.46	0.27	–0.24	0.01	0.01
σ	100	5.76	0.21	5.75	5.75	0.22	5.25	6.35	1.10	0.17	–0.15	0.02
ρ	100	0.73	0.11	0.73	0.74	0.10	0.44	0.97	0.54	–0.23	0.02	0.01
Nonlinear least squares estimator												
α	100	0.05	0.00	0.05	0.05	0.00	0.05	0.05	0.00	0.25	0.15	0.00
β	100	0.13	0.12	0.08	0.11	0.09	0.00	0.65	0.65	1.40	2.27	0.01
γ	100	0.22	0.27	0.10	0.16	0.13	0.00	1.66	1.66	2.24	6.82	0.03
σ	100	4.71	3.23	3.96	4.27	2.39	0.55	13.23	12.68	1.06	0.10	0.32
ρ	100	0.55	0.53	0,34	0.46	0.31	0.07	3.47	3.39	2.23	7.48	0.05
Parameter cascade method estimator												
α	100	0.05	0.00	0.05	0.05	0.00	0.05	0.05	0.00	–0.11	0.31	0.00
β	100	0.04	0.00	0.05	0.05	0.00	0.03	0.05	0.02	–1.59	1.91	0.00
γ	100	0.33	0.05	0.33	0.33	0.04	0.19	0.45	0.26	–0.21	0.09	0.01
σ	100	5.76	0.21	5.75	5.75	0.22	5.25	6.35	1.10	0.17	–0.15	0.02
ρ	100	0.74	0.11	0.73	0.74	0.10	0.44	0.97	0.53	–0.24	0.04	0.01

Table 2

Estimates of the cycle period obtained using the values of the simplified estimator, the NLS estimator, and the parameter cascade method estimator.
The true value of the cycle period is $T_G = 28.0993$

Estimation method	n	Mean	Standard deviation	Median	Trimmed mean	MAD	Min	Max	Range	Skew-ness	Kur-tosis	Stan-dard error
Simplified parameter estimation	100	36.75	2.97	36.45	36.5	2.99	30.7	46.87	16.18	0.91	1.22	0.3
Nonlinear least squares estimation	100	44.25	14	46.14	43.33	2.76	21.8	128.67	106.87	2.58	13.08	1.4
Parameter cascade method estimation	100	36.46	3.25	36.27	36.23	3.12	30.15	46.89	16.73	0.72	0.72	0.32

Table 3

Fixed-point estimates obtained using the values of the empirical estimator (mean values of u and v), the simplified estimator, the NLS estimator, and the parameter cascade method estimator. The true coordinates of the fixed point in the (u, v) subsystem are $(u^*, v^*) = (0.5, 0.5)$

Coordinate of variable	n	Mean	Standard deviation	Median	Trimmed	MAD	Min	Max	Range	Skew	Kurtosis	Standard error
Fixed point estimate using: empirical method												
u	100	0.48	0.00	0.48	0.48	0.00	0.46	0.49	0.02	–0.09	–0.13	0.00
v	100	0.50	0.00	0.50	0.50	0.00	0.49	0.51	0.02	–0.39	–0.07	0.00
Fixed point estimate using: simplified parameter estimator												
u	100	0.44	0.02	0.44	0.44	0.02	0.38	0.49	0.10	–0.26	–0.16	0.00
v	100	0.53	0.01	0.53	0.53	0.01	0.50	0.55	0.04	0.18	–0.38	0.00
Fixed point estimate using: nonlinear least squares estimator												
u	100	0.47	0.04	0.47	0.47	0.03	0.34	0.67	0.33	1.05	7.81	0.00
v	100	0.47	0.04	0.46	0.46	0.03	0.40	0.67	0.27	2.28	7.68	0.00
Fixed point estimate using: parameter cascade method estimator												
u	100	0.46	0.03	0.45	0.46	0.03	0.39	0.53	0.15	0.49	0.08	0.00
v	100	0.52	0.01	0.52	0.52	0.01	0.50	0.55	0.05	0.54	0.61	0.00

5.7. Finite-sample properties of the considered estimators of the Goodwin ODE model system observed in the simulation

Taking into account the simulation results in a finite sample, all three considered estimators of the structural parameters of the Goodwin model ODE system, i.e. the simplified estimator, the NLS estimator, and the parameter cascade estimator, exhibit bias relative to the known parameters of the data-generating process (DGP) of the simulation (see Tables 1–3 and the boxplots for particular estimator values: Figures 3, 5 and 11). Moreover, the iterative procedure of the NLS estimator, for a fraction of simulated time series, tends to converge to parameter values distant from the "real" DGP values. This is clearly visible in the (u, v) phase portrait of trajectories, with parameter values equal to the NLS estimates (see Figure 4). Some problems in recovering the cycle period T_G (overestimation) and the stable fixed point also appear for the considered estimators (see Table 3).

5.8. Empirical study: Goodwin model for the French economy (1962–2009)

Empirical data for France for the period 1962–2009 $T = 48$, extracted from the paper of Grasselli and Maheshwari (Grasselli, Maheshwari 2018), include:

- a: labor productivity,
- n: labor force,
- u: labor income share,
- v: employment rate.

Time series plots for a, n, u, v for France over the period 1962–2009 are shown in Figure 12.

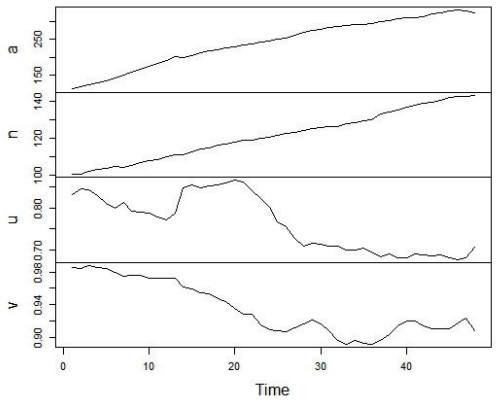

Figure 12. Time series, where u is labor income share, and v is employment rate. Data for France (1962–2009, $T = 48$)

Phase plot in the (u, v) space for empirical annual data for France for the period 1962–2009 ($T = 48$) is shown in Figure 13.

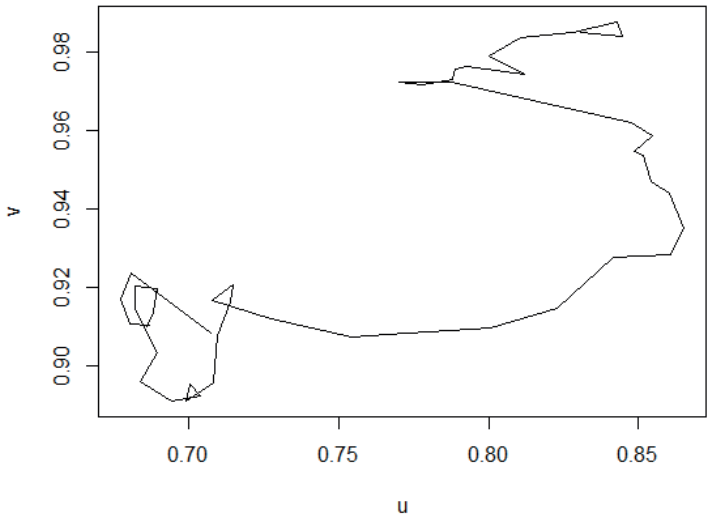

Figure 13. Phase portrait for (u, v), where u is labor income share, and v is employment rate.
Annual data for France (1962–2009, $T = 48$)

5.9. Simplified estimator of the four-equation ODE system (for *a*, *n*, *u*, *v*) of the Goodwin model for the French data (1962–2009)

First, we estimated the four-equation ODE system for the a, n, u and v state variables. We present results of the simplified estimator values for the structural and nuisance parameters for the four-equation ODE system for a, u and v, with the $iiN(0, \sigma_j)$, $j = a, u, v$, measurement noise process assumed for each of the state variables.

Values of the simplified estimator of the parameters of the four-equation ODE system of the Goodwin model, based on annual data for a, n, u, v for France for the period of 1962–2009 are:

$$\tilde{\alpha} = 0.0223,\ \tilde{\beta} = 0.0075,\ \tilde{\gamma} = 0.0738,\ \tilde{\sigma} = 8.9867,\ \tilde{\rho} = 0.0991.$$

The period of oscillation from the simplified estimates is $\tilde{T}_G = 70.9946$ years.

The fixed point, which is the center of the periodic ellipsoid from the simplified parameter estimates $(\tilde{u}^*, \tilde{v}^*) = (0.7325, 0.9698)$.

The time series plot for u and v with predicted values from the Goodwin model, based on the four-equation ODE model for the state variables a, n, u and v is shown in Figure 14.

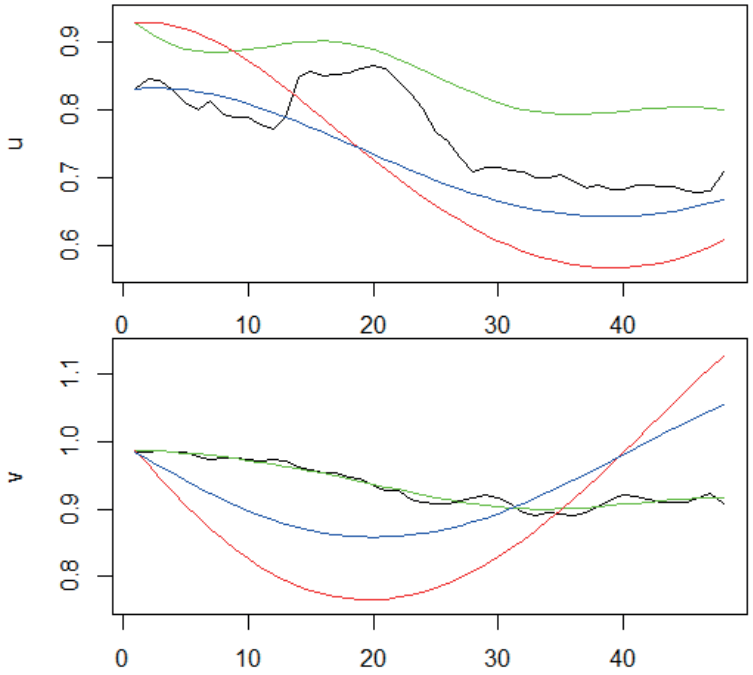

Figure 14. Time series plot of predicted values of the trajectories of u and v of the length $T = 48$, from the four-equation ODE system with structural parameter values obtained using the simplified estimator for the French data over the period 1962–2009, i.e. $\tilde{\alpha} = 0.0223$, $\tilde{\beta} = 0.0075$, $\tilde{\gamma} = 0.0738$, $\tilde{\sigma} = 8.9867$, $\tilde{\rho} = 0.0991$. The black curves represent empirical time series, while blue and red curves represent trajectories from the considered ODE model starting, respectively, from the empirically observed and (weighted by nuisance parameters in the B-spline basis) smoothed trajectories (green curves) as initial values of u and v; see below in the parameter cascade estimation subparagraph

The phase portrait for (u, v) with the ellipsoid and its center, based on the initial simplified estimates of the Goodwin model parameters (four-equation ODE model for the state variables a, n, u, v), is shown in Figure 15.

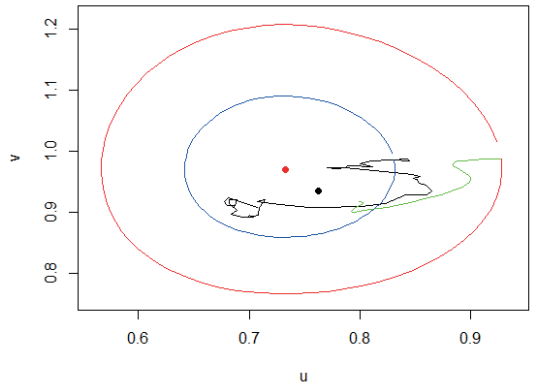

Figure 15. Phase plots in the (u, v) space of predicted trajectories of u and v, of length $T = [\hat{T}_G] = 71$ years, from the four-equation ODE system with structural parameter values obtained using the simplified estimator for the French data over the period 1962–2009, i.e. $\tilde{\alpha} = 0.0223$, $\tilde{\beta} = 0.0075$, $\tilde{\gamma} = 0.0738$, $\tilde{\sigma} = 8.9867$, $\tilde{\rho} = 0.0991$. The black curves represent empirical time series, while blue and red curves represent trajectories from the considered ODE model starting, respectively, from the empirically observed and (weighted by nuisance parameters in the B-spline basis) smoothed trajectories (green curve) as initial values of u and v; see below in the parameter cascade estimation subparagraph. The black point represents the empirically estimated fixed point of the (u, v) subsystem. The red point represents fixed point $(\tilde{u}^*, \tilde{v}^*) = (0.7325, 0.9698)$, calculated using simplified estimates of the structural parameters

5.10. Parameter cascade method estimator of the three-equation ODE system for (a, u, v) of the Goodwin model for the French data (1962–2009)

Here we estimate, using the parameter cascade method, the structural parameters of the Goodwin three-equation ODE system for the state variables a, u and v. The model assumes $iiN(0, \sigma_j)$, $j = a, u, v$ additive noise for each state variable. By experiment, we choose the B-spline functional basis of order 4 and with 4 knots.

We use the previously obtained simplified estimates as starting values for the iterative parameter cascade method procedure:

$$(\tilde{\alpha}_{(init)}, \tilde{\beta}_{(init)}, \tilde{\gamma}_{(init)}, \tilde{\sigma}_{(init)}, \tilde{\rho}_{(init)}) = (0.0223, 0.0075, 0.0738., 8.9867, 0.0991)$$

We use a multiple cross-validation procedure for selecting the hyperparameter λ (joint for all state variables) for the inner criterion of the parameter cascade method.

We assume a grid of λ values of 10^k, for integer values $k = -5, -4, \ldots, 4, 5$.

The value $\lambda = 10^{-2}$ was selected.

Parameter cascade method estimation of the ODE system's structural parameters and nuisance parameters was implemented in the `R` package `pCODE`. The package has been used in this study.

Table 4 presents the results of the parameter cascade method estimation (with $\lambda = 10^{-2}$) for the three-equation ODE system for a, n and v, with the $iiN(0, \sigma_j)$, $j = a, u, v$, measurement noise process assumed for each state variable.

Table 4

Values of the parameter cascade method estimator for structural parameters of the three-equation ODE system and nuisance parameters of the Goodwin model, based on annual data for a, n and v for France over the period 1962–2009

Structural parameters estimates: parameter cascade method									
α		β		γ		σ		ρ	
0.0125		0.0104		0.0761		8.9866		0.0921	
Nuisance parameters estimates: parameter cascade method									
1	2	3	4	5	6	7	8	9	10
0.9644	1.3329	1.9823	2.5418	2.8845	2.9517	0.928	0.8444	0.9688	0.7425
11	12	13	14	15	16	17	18		
0.8151	0.799	0.9875	0.9845	0.9566	0.8779	0.9164	0.917		

The following step is the estimation of variances of the parameter cascade method estimators for the structural parameters of the considered ODE system. To do this, we use the bootstrap method with 50 replications and $\lambda = 10^{-2}$. We also calculate the standard errors of the parameter cascade method estimators as square roots of bootstrap variances (see Table 5).

Table 5

Bootstrap variance and standard error estimates for the parameter cascade method estimators of the structural parameters of the considered three-equation ODE system of the Goodwin model

Statistic	α	β	γ	σ	ρ
Bootstrap variance estimator	9,2328E-05	1,8603E-05	1,8146E-05	7,1171E-06	1,6187E-05
Bootstrap standard errors of variance estimator	0.0096	0.0043	0.0043	0.0027	0.0040

The period of oscillations from the parameter cascade estimates is $\hat{T}_{period} = 71.0001$ years.

The fixed point, which is the center of the periodic ellipsoid from the parameter cascade method estimates, is $(\hat{u}^*, \hat{v}^*) = (0.7940, 0.9621)$.

The time series plot of the predicted values for the model based on the parameter cascade method estimates of the Goodwin model parameters (based on the three-equation ODE model for the state variables a, n and v) is shown in Figure 16.

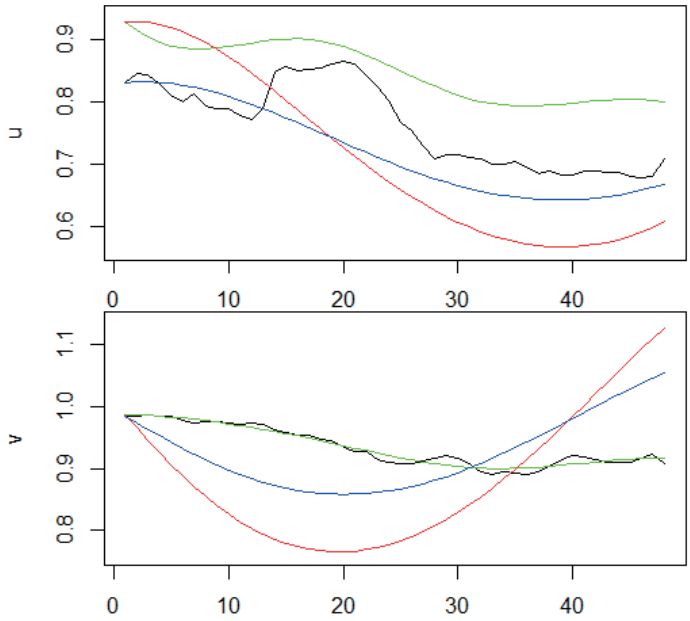

Figure 16. Time series plot of predicted trajectories of u and v, of length $T = 48$, from the three-equation ODE system with structural parameter values estimated using the parameter cascade method for the French data over the period 1962–2009 (with $\lambda = 10^{-2}$ and B-spline bases of order 4 with 4 knots), where $\tilde{\alpha} = 0.0125$, $\tilde{\beta} = 0.0104$, $\tilde{\gamma} = 0.0761$, $\tilde{\sigma} = 8.9866$, $\tilde{\rho} = 0.0921$. The black curves represent empirical time series, while blue and red curves represent trajectories from the considered ODE model starting, respectively, from the empirically observed and (weighted by nuisance parameters in the B-spline basis) smoothed initial values of u and v

The phase portrait showing the ellipsoid and its center, based on the parameter cascade method estimates of the Goodwin model parameters (three-equation ODE model for the state variables a, n and v), is shown in Figure 17.

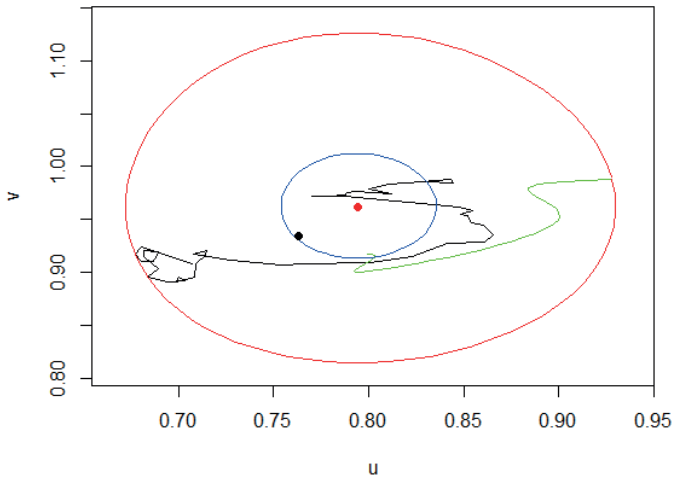

Figure 17. Phase plots in the (u, v) space of predicted trajectories of u and v, of length $T = [\hat{T}_G] = 72$ years, from the three-equation ODE system with structural parameter values estimated using the parameter cascade method for the French data over the period 1962–2009 (with $\lambda = 10^{-2}$ and B-spline bases of order 4 with 4 knots), where $\tilde{\alpha} = 0.0125$, $\tilde{\beta} = 0.0104$, $\tilde{\gamma} = 0.0761$, $\tilde{\sigma} = 8.9866$, $\tilde{\rho} = 0.0921$. The black curves represent empirical time series, while the blue and red curves represent trajectories from the considered ODE model starting, respectively, from the empirically observed and (weighted by nuisance parameters in the B-spline basis) smoothed initial values of u and v. The black point represents the empirically estimated fixed point of the (u, v) subsystem (computed as the mean of the empirical values). The red point represents the fixed-point estimate calculated using parameter cascade method estimators of the structural parameters: $(\hat{u}^*, \hat{v}^*) = (0.7940, 0.9621)$

5.11. NLS (nonlinear least squares) estimator of the three-equation ODE system for (a, u, v) of the Goodwin model for the French data over the period 1962–2009

Here we estimate the structural parameters of the Goodwin three-equation ODE system using the NLS method for the state variables a, u and v. The model assumes $iiN(0, \sigma_j)$, $j = a, u, v$ additive for each state variable.

We use the previously obtained simplified estimates as starting values for the iterative NLS procedure: $(\tilde{\alpha}_{(init)}, \tilde{\beta}_{(init)}, \tilde{\gamma}_{(init)}, \tilde{\sigma}_{(init)}, \tilde{\rho}_{(init)}) = (0.0223, 0.0075, 0.0738, 8.9867, 0.0991)$.

Table 6 presents the results of the NLS estimation of the structural parameters for the three-equation ODE system for a, u and v, with $iiN(0, \sigma_j)$, $j = a, u, v$, and a measurement noise process assumed for each state variable.

Table 6

Values of the NLS estimator for the structural parameters of the three-equation ODE system of the Goodwin model, based on annual data for a, u and v, for France over the period 1962–2009

Parameter	α	β	γ	σ	ρ
NLS estimate	0.0266	0.0247	0.0000	5.9879	0.0201

Based on the obtained NLS estimates of the Goodwin model's structural parameters, the requirement $\alpha + \gamma < \rho$ for economic meaningfulness is not satisfied.

Taking into account that the (u, v) trajectories based on the obtained NLS estimates lie outside of the square $[0,1]\times[0,1]$ most of the time, the NLS estimation results are not reliable.

5.12. Estimation results obtained using the parameter cascade and NLS methods in the empirical study

Values of the parameter cascade estimator for the Goodwin model, based on time series of length $T = 48$ for the French economy, imply a stable fixed point quite close to the empirical one. Plugging the estimated values into the (u, v) ODE subsystem results in a trajectory that remains within the rectangle $[0,1]\times[0,1]$ over the full cycle (estimated by the parameter cascade procedure as 72 years) when started from the empirically observed initial value. When the trajectory is started from the functionally smoothed initial value, it departs from the $[0,1]\times[0,1]$ box. Taking this into account and considering the simulation results presented earlier in this work, the empirical results should be interpreted with caution. Moreover, the available time series for France ($T = 48$) is too short to capture the full cycle ($\hat{T}_G = 48$), so it is not really possible to meaningfully recover the true length of the (u, v) cycle. There is also uncertainty as to whether the Goodwin model in its standard form can even approximately represent the true data-generating process for the considered state variables (model specification uncertainty).

For the NLS approach, the iterative procedure was stopped on the boundary of the feasible parameter set in the dimension of the γ parameter (which was zeroed). Based on the obtained NLS estimates, the requirement $\alpha + \gamma < \rho$ for economic meaningfulness is not satisfied. Problems with the NLS results also become apparent when examining trajectories in the (u, v) phase space, which, regardless of whether started from empirical or parameter cascade–smoothed initial values, remain outside the rectangle $[0,1]\times[0,1]$ during most of the cycle.

Other authors also report poor empirical outcomes in the estimation of structural parameters of the Goodwin model (including the recovery of the cycle length T_G and the fixed point). For example, Grasselli and Maheshwari consider estimators based on a discretization of the ODE system for time series data for 10 Western economies and report difficulties in recovering empirical fixed points as well as empirical trajectories of u and v (Grasselli, Maheshwari 2018). The authors suggest that the standard Goodwin system of ODEs should be respecified or extended to better match real macroeconomic data-generating processes. Similar empirical and theoretical conclusions regarding the estimation of the Goodwin model estimation can be found in literature (Harvie 2000; Basu 2024).

6. Conclusions

Our study of ODE systems with measurement error in economic modeling – focusing on the empirical Goodwin model – reveals both methodological progress and practical challenges. The parameter cascade method offers a functional (basis-function-expansion-driven) approach to handling measurement noise in ODE systems; however the results reveal limitations: in finite samples, all estimators exhibit bias, and the NLS method sometimes produces estimates far from the true parameters values, leading to unrealistic trajectories. The non-identifiability of the standard two-equation Goodwin model leads us to extend it with a third equation for labor productivity, introduced in the derivation of the Goodwin model from macroeconomic dynamic assumptions. Applying the model to French data, we struggle to recover realistic dynamics, which is in line with existing literature, suggesting the need to respecify or extend the model. Overall, while the parameter cascade and related methods are valuable tools for analyzing nonlinear systems with measurement error, their performance depends strongly on model structure, data quality, and underlying economic assumptions.

Acknowledgments

The contribution by Jerzy Rydlewski was supported by the AGH University of Krakow under grants no. 16.16.420.054 and 501.00 10000 2600, funded by the Polish Ministry of Science and Higher Education.

The contribution by Przemysław Jaśko presents the results of Project no. 059/EFP/2025/POT, financed from the subsidy granted to the Krakow University of Economics.

References

Basu, D. (2024) 'A note on the theory and empirics of the Goodwin model' , Working Paper, University of Massachusetts Amherst, https://hdl.handle.net/20.500.14394/54830.

Cao, J. and Ramsay, J.O. (2007) 'Parameter cascades and profiling in functional data analysis', *Computational Statistics*, vol. 22(3), pp. 335–351, https://doi.org/10.1007/s00180-007-0044-1.

Cao, J., Fussmann, G.F. and Ramsay, J.O. (2008) 'Estimating a predator-prey dynamical model with the parameter cascades method', *Biometrics*, vol. 64(3), pp. 959–967, https://doi.org/10.1111/j.1541-0420.2007.00942.x.

Cao, J., Huang, J.Z. and Wu, H. (2012) 'Penalized nonlinear least squares estimation of time-varying parameters in ordinary differential equations', *Journal of Computational and Graphical Statistics*, vol. 21(1), pp. 42–56, https://doi.org/10.1198/jcgs.2011.10021.

Denis-Vidal, L., Joly-Blanchard, G. and Noiret, C. (2003) 'System identifiability (symbolic computation) and parameter estimation (numerical computation)', *Numerical Algorithms*, vol. 34(2), pp. 283–292, https://doi.org/10.1023/B:NUMA.0000005366.05704.88.

Deuflhard, P. and Bornemann, F. (2012) Scientific computing with ordinary differential equations, New York: Springer.

Dong, R., Goodbrake, C., Harrington, H.A. and Pogudin, G. (2023) "Differential elimination for dynamical models via projections with applications to structural identifiability', *SIAM Journal on Applied Algebra and Geometry*, vol. 7(1), pp. 194–235, https://doi.org/10.1137/22M1469067.

Elzhov, T.V., Mullen, K.M., Spiess, A.N. and Bolker, B. (2026) Package 'minpack.lm'. May 8, 2026. R interface Levenberg-Marquardt nonlinear least-sq. Algorithm found MINPACK plus support bounds, [Online], Available: https://cran.r-project.org/web/packages/minpack.lm/minpack.lm.pdf [9 Jun 2026].

Grasselli, M.R. and Maheshwari, A. (2018) 'Testing a Goodwin model with general capital accumulation rate', *Metroeconomica*, vol. 69(3), pp. 619–643, https://doi.org/10.1111/meca.12204.

Goodwin, R.M. (1967) 'A growth cycle', in Feldstein, C.H. (ed.) Socialism, capitalism, and economic growth. Essays presented to Maurice Dobb, Cambridge: Cambridge University Press.

Guzowska, M. (2011) 'Dynamiczne własności dyskretnej wersji modelu wzrostu Goodwina', *Zeszyty Naukowe/Uniwersytet Ekonomiczny w Poznaniu*, vol. 211, pp. 125–136, [Online], Available: https://bazekon.uek.krakow.pl/zeszyty/171212829 [9 Jun 2026].

Harvie, D. (2000) 'Testing Goodwin: growth cycles in ten OECD countries', *Cambridge Journal of Economics*, vol. 24(3), pp. 349–376, https://doi.org/10.1093/cje/24.3.349.

Hong, H., Ovchinnikov, A., Pogudin, G. and Yap, C. (2020) 'Global identifiability of differential models', *Communications on Pure and Applied Mathematics*, vol. 73(9), pp. 1831–1879, https://doi.org/10.1002/cpa.21921.

Hooker, G., Ramsay, J.O. and Xiao, L. (2016) 'CollocInfer: Collocation inference in differential equation models', *Journal of Statistical Software*, vol. 75, pp. 1–52, https://doi.org/10.18637/jss.v075.i02.

Jakimowicz, A. (2003) Od Keynesa do teorii chaosu: ewolucja teorii wahań koniunkturalnych, Warszawa: Wydawnictwo Naukowe PWN.

Karami, H., Bleichrodt, A., Luo R. and Chowell, G. (2025). BayesianFitForecast: a user-friendly R toolbox for parameter estimation and forecasting with ordinary differential equations, *BMC Medical Informatics and Decision Making*, vol. 25, 385, https://doi.org/10.1186/s12911-025-03208-z.

Kolchin, E.R. (1973) Differential algebra & algebraic groups, New York–London: Academic Press.

Liyanage, Y.R., Saucedo, O., Tuncer, N. and Chowell, G. (2025) 'A tutorial on structural identifiability of epidemic models using *StructuralIdentifiability.Jl*', *arXiv preprint arXiv:2505.10517v5*, https://doi.org/10.48550/arXiv.2505.10517.

Park, S.W. (2022) Getting started with the fitode package, [Online], Available: https://r-mirror.zim.uni-due.de/web/packages/fitode/vignettes/fitode.pdf [9 Jun 2026].

Petzold, L. (1983) 'Automatic selection of methods for solving stiff and nonstiff systems of ordinary differential equations', *SIAM Journal on Scientific and Statistical Computing*, vol. 4(1), pp. 136–148, https://doi.org/10.1137/0904010.

Qi, X. and Zhao, H. (2010) 'Asymptotic efficiency and finite-sample properties of the generalized profiling estimation of parameters in ordinary differential equations', *Annals of Statistics*, vol. 38 (1) pp. 435–481, https://doi.org/10.1214/09-AOS724.

Ramsay, J.O., Hooker, G., Campbell, D. and Cao, J. (2007) 'Parameter estimation for differential equations: a generalized smoothing approach', *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 69(5), pp. 741–796, https://doi.org/10.1111/j.1467-9868.2007.00610.x.

Ramsay, J.O., Hooker, G. and Graves, S. (2009) Functional data analysis with R and MATLAB, New York: Springer.

Ramsay, J.O. and Silverman, B.W. (2005) Functional data analysis, New York: Springer .

Rydlewski, J.P. (2025) Wybrane zastosowania metod funkcjonalnej analizy danych w ekonomii, Kraków: Wydawnictwa AGH, https://doi.org/10.7494/978-83-68219-25-8.

Sedoglavic, A. (2002) 'A probabilistic algorithm to test local algebraic observability in polynomial time', *Journal of Symbolic Computation*, vol. 33, pp. 735–755, https://doi.org/10.1006/jsco.2002.0532.

Soetaert, K., Petzoldt, T. and Setzer, R.W. (2010) 'Solving differential equations in R: Package deSolve', *Journal of Statistical Software*, vol. 33(9), pp. 1–25, https://doi.org/10.18637/jss.v033.i09 [9 Jun 2026]

Varah, J.M. (1982) 'A spline least squares method for numerical parameter estimation in differential equations', *SIAM Journal on Scientific and Statistical Computing*, vol. 3(1), pp. 28–46, https://doi.org/10.1137/0903003.

Wang, H. and Cao, J. (2019) Parameter cascade method for ODE models with pCODE [Online], Available: https://cran.r-project.org/web/packages/pCODE/vignettes/pcode-vignette.html [9 Jun 2026].

Wang, H. and Cao, J. (2023) 'pCODE: Estimating parameters of ODE models', *The R Journal*, vol. 14(4), [Online], Available: https://journal.r-project.org/articles/RJ-2023-018/ [9 Jun 2026].

Summary

This study investigates the application of functional data analysis and the parameter cascade method to the estimation of structural parameters in the Goodwin model of economic growth cycles. The Goodwin model, formulated as a system of ordinary differential equations describing the dynamic interaction between labour's share of income and the employment rate, was initially estimated using functional data representations. However, the model did not yield satisfactory empirical results, which can be partly attributed to its structural non-identifiability and the limited time span of the available data, which do not cover a full business cycle. To address these limitations, an identifiable model based on the Volterra–Lotka system was introduced as an alternative specification. Using annual data for France over the period 1962–2009, both simulation and empirical analyses were conducted to assess the performance of the parameter cascade method relative to the nonlinear least squares estimator and simplified method estimator of the Goodwin model three equation ODE system. The results indicate that the functional-data-based approach, particularly the parameter cascade method, provides a promising framework for the estimation of continuous-time dynamic economic models with measurement error. Although the findings remain preliminary, they demonstrate the methodological potential of combining functional data analysis with ODE-based representations of macroeconomic processes.

JEL codes: JEL E32, C63, C13

Keywords: Goodwin model, ODE systems in economics, functional data analysis, collocation methods, parameter cascade method, France's business cycle

Received: 26.11.2025
Accepted: 26.05.2026