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Henryk Gurgul*, Artur Machno*, Roland Mestel**

Modeling of Returns and Trading Volume by Regime Switching Copulas

1. Introduction

The relations among international stock markets have been investigated in many papers, especially in the periods of financial crises. The topic under study is important for market participants because due to globalization process the global markets are becoming more and more dependent. This observation follows from liberalization and deregulation in money and capital markets. In addition, the globalization process diminishes opportunities for international diversification.

The financial data show asymmetric dependence. This feature is reflected in the observation that in the bear phase stock market data such as returns, trading volume and volatility are getting to be more dependent than in the bull stock market. This means that investors might lose advantages of international portfolios and these portfolios may be more risky than what the investors assume. The occurrence of such asymmetric interdependence is also probable between returns and trading volume. The goal of this article is to describe co-movements of realized volatility and trading volume for selected stocks listed on Vienna Stock Exchange.

The remainder of the contribution is organized in the following way: in section 2 we conduct the literature overview concerning the dependence concepts, including regime switching models and copulas and discuss the recent contributions to the subject; in section 3 we present the data; in the following section the dependence measures and copulas are overviewed; in the fifth section copula regime switching model is described; in the sixth section the results are reported and evaluated; section 7 concludes the paper.

* AGH University of Science and Technology in Cracow, Department of Applications of Mathematics in Economics, e–mails: henryk.gurgul@gmail.com; artur.machno@gmail.com
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** University of Applied Sciences Joanneum in Graz, Department of Banking and Insurance, e-mail: roland.mestel@uni-graz.at

2. Literature overview

The dependence between the stock markets can be measured through such variables as stock return, trading volume and volatility. The most frequently used methodology in the investigations of the interdependencies is based on Granger causality and VAR model (see [18]). In one of the earliest contributions on dependency of stock markets Eun and Shim ([15]) checked the relationships among nine major stock markets including Australia, Canada, France, Germany, Hong Kong, Japan, Switzerland, the UK and the US by means of the Vector Autoregressive (VAR) Model. The authors found that news in the US market has a major impact on the other markets. Lin *et al.* ([31]) focused on interdependence between the returns and volatility of Japan and the US market indices using high frequency data of daytime and overnight returns. They established that daytime returns in the US or Japan market were linked with the overnight returns in the other.

Kim and Rogers ([27]) studied the dynamic interdependence between the stock markets of Korea, Japan, and the US. They underlined the importance of Japanese and the US stock markets for Korean market since the last became more open for foreign investors. By mean of EGARCH model Booth *et al.* ([11]) found strong interdependence among the Danish, Finnish, Norwegian and Swedish Stock Market. According to the authors the essential dependence started with the so-called Thailand currency crisis. However, it was not observed after the Hong Kong crisis. Ng ([33]) established significant causality running from the US and Japan stock market to six Asian markets: Hong Kong, Korea, Malaysia, Singapore, Taiwan and Thailand. Klein *et al.* ([28]) by means of wavelets technique, applied to three developed markets: US, Germany and Japan and two emerging markets Egypt and Turkey proved that changes in these developed markets had effects on the emerging markets. In the paper [6], using the VAR-EGARCH model, it is checked the interdependence among three EU markets namely Germany, France and the UK. The results supported the hypothesis of the cointegration among the mentioned stock markets.

Sharkasi et al. ([42]) used wavelet analysis and found the global co-movements among seven stock markets, three in Europe (Irish, UK, and Portuguese), two in the Americas (namely US, and Brazilian) and two in Asia (Japanese and Hong Kong).

The contributions by Ammermann and Patterson ([2]), Lim et al. (30), Lim and Hinich ([29]), Bessler et al. ([7]) or Bonilla et al. ([10]) tried to established a different pattern of the stock price development. The authors detected long random walk phases. They alternated with short ones and showed significant linear and/or nonlinear correlations. The contributors thought that these serial dependencies had an episodic character. Due to these contributors the serial dependencies caused the low performance of the forecasting models. Nivet ([34])

checked the random walk hypothesis for the Warsaw Stock Exchange. Worthington and Higgs ([46]) proved the efficiency on the Hungarian, Polish, Czech and Russian stock markets. The Hungarian stock market followed the random walk. Gilmore and McManus ([17]) found autocorrelations in some of the Central and Eastern European stock markets. Schotman and Zalewska ([43]), claimed that the nonsynchronous trading and asymmetric response to good and bad news were reason for autocorrelation.

Todea et al. ([45]) applied the Hinich–Patterson windowed-test procedure. By means of it, he investigated the temporal persistence of linear and, especially, nonlinear interdependencies among six Central and Eastern European stock markets.

Issues concerning asymmetry of dependence, were analyzed by Longin and Solnik ([32]). The contributors found (by means of the constant conditional correlation (CCC) model introduced by Bollerslev ([8]) that the correlations between the stock markets over a period of three decades were not stable over this time period. In addition, they increased during more volatile periods. Moreover, they depended on some economic variables such as interest rates, buybacks or dividend yields. Some results based on extreme value theory were showed in Ang and Chen ([5]). The authors derived a test for asymmetric correlation. They suggested comparison of empirical and model-based conditional correlations. The authors justified that regime switching models were most suitable for modelling of asymmetry. Ang and Bekaert in [3] and [4] applied a Gaussian Markov switching model for international returns. They estimated two regimes: a bull regime with positive mean, low volatilities and low correlations; and a bear regime with negative returns, high volatilities and correlation.

Regime switching models were introduced in econometrics by Hamilton ([21]). Currently they found many applications in finance. In the papers [19] and [20] is being concerned the interest rate, the methodology of regime switching models was used. The contributors used also a regime switching model for international financial returns. In the paper ([31]) the regime switching modelling was applied to the model correlation. The author assumed a normal distribution. The marginals were modelled with the GARCH. The model by Pelletier was something “intermediate” between the constant conditional correlation (CCC) of Bollerslev ([8]) and the dynamic conditional correlation (DCC) model of Engle ([14]).

Patton ([36]) indicated a significant asymmetry in the structure of dependence of the financial returns what is very important for a certain kind of investors. In his further contributions (see [37] and [38]), he introduced conditional copulas and time-varying models of bivariate dependence of coefficients in order to model foreign exchange rates. Jondeau and Rockinger ([26]) applied the skewed-*t* GARCH models for returns with univariate time-varying skewness. Finally, in order to measure the dependence between pairs of countries, they used a time-varying or a switching Gaussian, or Student *t* copula. Jondeau and Rockinger ([26]) and Hu ([25]) suggested the so-called copula based multivariate dynamic (CMD) model. Klein at al. ([28]) conducted an extensive simulation study, and demonstrated that CMD models were proper tools for investigating different time series with the GARCH structure for the squared residuals. They showed that the copula (mis-) specification should play a key role before the usage of the CMD model.

In recent years the economists combined copulas and regime switching models in bivariate financial data. Rodriguez ([41]) and Okimoto ([35]) applied regime switching copulas for pairs of international stock indices. While Okimoto ([35]) dealt with the US-UK pair, Rodriguez ([41]) worked with pairs of Latin American and Asian countries. The contributors applied methodology developed by Ramchand and Susmel ([40]) with a structure where variances, means and correlations switched together in the two-variable system. Garcia and Tsafak ([16]) estimated a regime switching model in a four-variable system of domestic and foreign stocks and bonds. The authors used a mixture of bivariate copulas to model the dependence between all possible pairs of variables.

Chollete et al. ([12]) generalized the Pelletier ([39]) model to the non-Gaussian case. The authors excluded the Gaussian assumption, because the returns were not Gaussian and suggested a regime switching structure for dependence. They used flexible multivariate copulas. They also tried to separate the asymmetry in the marginals from the one in the dependence. This could not have been done in a Gaussian switching model. Their investigations were based on copulas. The authors instead of Gaussian marginal distribution applied the skewed *t* GARCH model of Hansen ([24]).

The authors applied their model in multivariate case and therefore they made their approach more appropriate for practical applications. They used the canonical vine copula, which allowed very general types of dependence.

The authors found that the VaR and Expected Shortfall of the canonical vine models were essentially better than the Student *t* or Gaussian copula models, which implied that the inappropriate usage of the latter models could lead to the underestimation of the risk of a portfolio.

In order to model the observed asymmetric dependence in pairs trading volume realized volatility, we estimated a bivariate copula based on a regime switching model. We applied this model to high frequency data from Vienna Stock Exchange. The choice of copula is important for the risk management, because it modifies the Value at Risk (VaR) and Expected Shortfall of international portfolio returns. We will check the dynamics of the interdependence between realized volatility and trading volume. The main goal is to document changes in the dependency and the asymmetry in both quiet and hectic (bull or bear) phase in the stock markets.

3. Data

The database consists of *tick-by-tick* prices of five stocks of Austrian Trade index. In particular, the data set consists of stock prices of Andritz, ERSTE, OMV, TKA, and Voest from January 2, 2006 through November 9, 2011. Volatility is measured by realized variance computed as the sum of the intraday squared log returns. Therefore, the following formula presents the realized volatility *RV*,

$$RV_t = \sum_{i=1}^{M} r_{t_i}^2, \;\; t = 1, \ldots, T,$$

where r_{t_i} are intraday log returns and M is the number of intraday observations. It has been analyzed and solved that prices sampled at high frequency are affected by microstructure noise. This phenomenon has been solved in various ways (see [26] and [6]). To avoid this effect, the simplest way is to sample at a lower frequencies (see [9] and [13]). In this paper, we use 5-minutes transaction prices. The set of time series analyzed in this paper are created as follows: firstly, returns over five minutes intervals were calculated and realized volatility was calculated as the sum of squared returns. Daily volume was obtained as the sum of intraday volume. Logarithms of realized volatility and trading volume series appeared to be modelled better by the method presented in this article. Therefore, a realized volatility and a trading volume series is understood as the logarithm of the corresponding time series. Volatility and volume series consist of 1454 observations for all five stocks.

The following two tables (Table 1, 2) present summary statistics of examined time series.

Table 1

Realized volatility time series summary statistics

	Andritz	Erste	OMV	TKA	Voest
Mean	–7.7657	–7.8810	–8.0110	–8.1135	–7.6640
Median	–7.8396	–8.0704	–8.1167	–8.2144	–7.7764
Variance	0.8848	1.1065	0.7354	0.8181	0.7760
Q_1	–8.4043	–8.6427	–8.5739	–8.7440	–8.2992
Q_3	–7.2397	–7.2471	–7.5862	–7.6103	–7.0914
Skewness	0.5612	0.5650	0.7386	0.5959	0.5016

Table 2

Daily volume time series summary statistics

	Andritz	Erste	OMV	TKA	Voest
Mean	11.9151	14.4443	13.3927	13.8826	13.6980
Median	11.8941	14.4065	13.3850	13.8908	13.5845
Variance	0.3242	0.7770	0.2392	0.2618	0.7029
Q_1	11.5349	13.7355	13.0620	13.5338	13.0652
Q_3	12.3003	15.1522	13.7069	14.2245	14.3680
Skewness	0.1462	0.0653	0.2342	0.0844	0.2119

4. Copulas and dependence measures

In this article, we deal with a dependence between two variables: a realized volatility and a daily trading volume by copulas. Before doing this we report briefly some definitions and properties of copulas. Most of definitions and some of properties can be extended to the multivariate case, especially the central result of copula theory which is Sklar's theorem, expressed and proved by Sklar ([44]). It states that a joint distribution can be decomposed into marginal distributions and a copula. It also gives a simple way to create bivariate distribution from any given marginal distributions. For bivariate cumulative distribution function $H(x,y)$ and its margins $F(x)$ and $G(y)$, according to Sklar's theorem, there exists a function $C:[0,1]^2 \to [0,1]$, called copula, such that $H(x, y) = C(F(x), G(y))$. In the case of continuous variables, the function is unique and is equal to $C(u,v) = H(F^{-1}(u), G^{-1}(v))$, for $u, v \in [0,1]$.

Conversely, every function $C:[0,1]^2 \to [0,1]$ which has the following properties:

1. For every $u, v \in [0,1]$, $C(u,0) = C(0,v) = 0$,

$$C(u,1) = u \text{ and } C(1,v) = v. \tag{1}$$

2. For every $u_1,u_2,v_1,v_2 \in [0,1]$, such that $u_1 \le u_2$ and $v_1 \le v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0. \tag{2}$$

is called copula.

Those two concepts of copula function are in fact equivalent. Every copula obtained from Sklar's theorem satisfies (1) and (2). Conversely, if $C:[0,1]^2 \to [0,1]$ satisfies (1) and (2), there exists a pair of variables X and Y for which C is the copula obtained from Sklar's theorem. Clearly, a copula can be considered as the cumulative distribution function of the bivariate random variable with uniformly distributed margins. According to this, for given copula C, we denote its density function as c. Therefore, for continuous random variables X and Y:

$$h(x,y) = f(x)g(y)c\big(F(x),G(y)\big),$$

where $h(x,y)$ is joint density function of random vector (X,Y); $f(x)$ and $g(y)$ are density functions and $F(x)$ and $F(y)$ are cumulative distribution functions of X and Y, respectively.

The simplest copula is $\Pi(u,v) = uv$. It connects independent margins. Other interesting examples are, so called, Fréchet-Hoeffding copula bounds, for u, v in $[0,1]$. They are defined by:

$$W(u,v) = \max\{u+v-1,0\},\ M(u,v) = \min\{u,v\}.$$

The generalization of the function W to higher dimensions is not a copula. Only in two dimensions it is a copula, in which case it corresponds to countermonotonic random variables.

The function W and M are called lower and upper Fréchet-Hoeffding copula bounds, because for any copula C and u, v in $[0,1]$, we have:

$$W(u,v) \le C(u,v) \le M(u,v).$$

Copulas used in this article are the Gaussian copula and two-parameter Archimedean copulas BB1, BB4 and BB7. The Gaussian copula is constructed from a bivariate normal distribution. For given correlation ρ, the Gaussian copula with parameter ρ can be written as

$$C_\rho^{Gauss} = \Phi_\Sigma\left(\Phi^{-1}(u),\Phi^{-1}(v)\right), \tag{3}$$

where Φ^{-1} is the inverse cumulative distribution function of the standard normal distribution and Φ_Σ is the joint cumulative distribution function of a bivariate normal distribution with mean zero and covariance matrix $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

Below BB1, BB4 and BB7 families are defined:

$$C_{\theta,\delta}^{BB1} = \left\{1 + \left[\left(u^{-\theta} - 1\right)^{\delta} + \left(v^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}}, \theta \geq 0, \delta \geq 1; \tag{4}$$

$$C_{\theta,\delta}^{BB4} = \left\{u^{-\theta} + v^{-\theta} - 1 + \left[\left(u^{-\theta} - 1\right)^{-\delta} + \left(v^{-\theta} - 1\right)^{-\delta}\right]^{-\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}}, \theta \geq 0, \delta > 0; \tag{5}$$

$$C_{\theta,\delta}^{BB7} = 1 - \left\{1 - \left[\left(1 - \bar{u}^{\theta}\right)^{-\delta} + \left(1 - \bar{v}^{\theta}\right)^{-\delta}\right]^{-\frac{1}{\delta}}\right\}^{\frac{1}{\theta}}, \theta \geq 1, \delta \geq 0; \tag{6}$$

where $\bar{u} := 1 - u$ and $\bar{v} := 1 - v$.

Due to Sklar's theorem, copula function is a very efficient tool to study the structure of dependence of multivariate random variables. In recent years copulas are widely used to describe the structure of dependence between financial variables. The most traditional dependence measure is Pearson correlation. However, it measures only linear dependence and works only in the range of the spherical and elliptical distributions. The exceedance correlation is generalized Pearson coefficient which measures asymmetric dependence. It is defined as the correlation between two variables, conditional on both variables being below or above some fixed levels. Lower exceedance correlation between variables X and Y is defined as:

$$ecorr_{\theta_1\theta_2}^{L}\left(X,Y\right) = corr\left(X,Y \middle| X \leq \theta_1, Y \leq \theta_2\right),$$

where θ_1 and θ_2 are fixed levels. Analogously, for fixed levels θ_1 and θ_2, upper exceedance correlation between variables X and Y is defined as:

$$ecorr_{\theta_1\theta_2}^{U}\left(X,Y\right) := corr\left(X,Y \middle| X \geq \theta_1, Y \geq \theta_2\right).$$

Exceedance correlation is used particularly in risk management, where negative extreme values of an investment return are crucial. The main problem with exceedance correlation is that it is dependent on margins. Another weakness is that it is computed only from observations which are below (above) the fixed limit. Therefore, as the limit is further out into the tail as exceedance correlation is computed less precisely.

Another tail dependence measure is quantile dependence. For random variables X and Y with distribution functions F and G, respectively, the lower tail dependence at threshold α is defined as $P[Y < G^{-1}(\alpha) \mid X < F^{-1}(\alpha)]$. Analogously, the upper tail dependence at threshold α is defined as $[Y > G^{-1}(\alpha) \mid X > F^{-1}(\alpha)]$. The dependence measure which is particularly interesting is tail dependence obtained as the limit of quantile dependence. We define lower tail dependence λ_L of X and Y as:

$$\lambda_L = \lim_{\alpha \to 0^+} P\left[Y < G^{-1}(\alpha) | X < F^{-1}(\alpha)\right],$$

and upper tail dependence λ_U of X and Y as:

$$\lambda_U = \lim_{\alpha \to 1^-} P\left[Y > G^{-1}(\alpha) | X > F^{-1}(\alpha)\right].$$

Variables X and Y are called asymptotically dependent if $\lambda_L \in (0,1]$ and asymptotically independent if $\lambda_L = 0$. Unlike exceedance correlations, tail dependence is independent of margins. Let C be the copula obtained from Sklar's theorem for continuous random variables X and Y. In this case, lower tail dependence λ_L and upper tail dependence λ_U can be computed as follows:

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}, \ \lambda_U = \lim_{u \to 1^-} \frac{C(u,u)}{u}. \tag{7}$$

Another class of dependent measures are measures based on ranks of variables. The two most popular rank correlations are Kendall's τ and Spearman's ρ. Both rely on the notion of concordance. Let (x_1, y_1) and (x_2, y_2) be two observations of random vector (X, Y). We say that the pair is *concordant* whenever $(y_1 - y_2)(x_1 - x_2) > 0$, and *discordant* whenever $(y_1 - y_2)(x_1 - x_2) > 0$. Intuitively, a pair of random variables are concordant if large values of one variable occur more likely with large values of the other variable. For random variables X and Y, Kendall's τ is defined as:

$$\tau = P\left[(y_1 - y_2)(x_1 - x_2) > 0\right] - P\left[(y_1 - y_2)(x_1 - x_2) < 0\right],$$

where (x_1, y_1) and (x_2, y_2) are independent observations of (X, Y). In terms of copulas, Kendall's τ has concise form. For the pair of random variables X and Y and its copula C, we have:

$$\tau_C = 4 \int_{[0,1]^2} C(u,v)\, dC(u,v) - 1. \tag{8}$$

Since copula is invariant with respect to any monotonic transformation, Kendall's τ has also this property. From the formula (8) we see that Kendall's τ does not depend on marginal distributions.

We may think of copulas as of the function describing the structure of dependence of two variables. The copula for a pair of random variables may be used as an efficient tool for the dependence investigation. It provides us with simple formulas of dependency measures such as tail dependence or Kendall's τ. The Table 3 presents lower and upper tail dependencies, defined by (7), for copulas presented before.

Table 3

Tail dependencies for Gaussian, BB1, BB2, BB4 and BB7 copulas

	λ_L	λ_U
C_ρ^{Gauss}	0	0
$C_{\theta,\delta}^{\text{BB1}}$	$2^{-\frac{1}{\delta\theta}}$	$2-2^{\frac{1}{\theta}}$
$C_{\theta,\delta}^{\text{BB4}}$	$(2-2^{-\frac{1}{\delta}})^{-\frac{1}{\theta}}$	$2^{-\frac{1}{\delta}}$
$C_{\theta,\delta}^{\text{BB7}}$	$2^{-\frac{1}{\delta}}$	$2-2^{\frac{1}{\theta}}$

The Table 4 presents empirical dependence measures for analyzed pairs of time series.

Table 4

Realized volatility time series summary statistics

	Andritz	**Erste**	**OMV**	**TKA**	**Voest**
ρ	0.5621	0.4052	0.4320	0.4260	0.1430
Kendall's *t*	0.3794	0.2696	0.2786	0.2961	0.1129
λ_L	0.2945	0.2123	0.3151	0.3219	0.2671
λ_U	0.4452	0.3493	0.3082	0.2260	0.0548
$ecorr^{L}_{Q_1^1,Q_1^2}$	0.0193	0.1927	0.1279	0.1776	0.2630
$ecorr^{U}_{Q_3^1,Q_3^2}$	0.3941	0.2282	0.1391	0.0398	-0.2493

Here ρ is Pearson's correlation, Q_α^1 and Q_α^2 are α-quantiles of a realized volatility series and a daily volume series, respectively. Tail dependencies λ_L and λ_U are approximated by $P\left[Y < G^{-1}(0.1) | X < F^{-1}(0.1)\right]$ and $P\left[Y > G^{-1}(0.9) | X > F^{-1}(0.9)\right]$, respectively.

5. Regime switching copula model

Switching models have recently been broadly explored. These models were firstly presented by Hamilton (1989) and widely analyzed by Hamilton (1994). In this article, a switching model based on copulas is presented and used to investigate relation between volatility and trading volume on a stock market. The model is based on two-state Markov process. Let $y_t = (y_{1t}, y_{2t})$ be the vector of the realized volatility and the daily trading volume, and let $Y_t = (y_t, y_{t-1}, y_{t-2}, \ldots)$ be a series of observations available at time t.

We denote the state process by s_t. Joint density function f for y_t is defined as:

$$f\left(y_t | Y_{t-1}, s_t = j\right) = c^{(j)}\left(F_1(y_{1t};\delta_1), F_2(y_{2t};\delta_2)\right) \cdot f_1(y_{1t};\delta_1) \cdot f_2(y_{2t};\delta_2),$$

where F_i and f_i, for $i = 1,2$, are marginal distribution function and density of y_i, and δ_i is a parameter vector for the marginal distribution. The copula $c^{(1)}$ in the first regime is chosen in order to model an asymmetry in tails of the count distribution. Precisely, $c^{(1)}$ is chosen as a one of three copulas BB1, BB4 and BB7 defined by (4), (5) and (6). Conversely, the copula $c^{(2)}$ in the second regime is symmetric Gaussian copula defined by (5). The probability that the state i precedes the state j is denoted by $p_{ij} = P[s_t = j \mid s_{t-1} = i]$. All four probabilities form transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}. \tag{9}$$

The estimation of regime switching copula model is based on the maximum likelihood estimation. Unfortunately, computing power needed to maximize likelihood function is enormous. To simplify calculation decomposition of likelihood function to margins likelihood functions and the copula likelihood function is made. Formally, for $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_T)$ log likelihood function is defined by:

$$L\left(\boldsymbol{Y};\delta,\theta\right) = \sum_{t=1}^{T} \ln f\left(y_t \mid Y_{t-1};\delta,\theta\right),$$

and it is decomposed to L_m and L_c such that:

$$L\left(\boldsymbol{Y};\delta,\theta\right) = L_m\left(\boldsymbol{Y};\delta\right) + L_c\left(\boldsymbol{Y};\delta,\theta\right),$$

where

$$L_m\left(\boldsymbol{Y};\delta\right) = \sum_{t=1}^{T} \left[\ln f_1\left(y_{1t} \mid (Y_{1t-1};\delta_1\right) + \ln f_2\left(y_{2t} | \left(Y_{2t-1};\delta_2\right)\right)\right], \tag{10}$$

$$L_c\left(\boldsymbol{Y};\delta,\theta\right)=\sum_{t=1}^{T}\ln c[F_2(y_{1t}\mid(Y_{1t-1};\delta_1),F_2\left(y_{2t}\middle|(Y_{2t-1};\delta_2\right);\theta\big]. \tag{11}$$

The estimation is conducted in two steps. Firstly, estimations of parameters δ_1 and δ_2 of marginal distribution is performed by the maximization of the likelihood function defined by (10). Secondly, we maximize the likelihood function defined by (11) to estimate parameters θ_1 and θ_2 of copulas $c^{(1)}$ and $c^{(2)}$, and transition matrix given by (9).

A method of estimation of marginal distributions depends on the model which we choose to describe volatility and volume series. To do the second part of estimation of regime switching copula model we use Hamilton filter defined by the following recurrence relations:

$$\hat{\xi}_{t|t}=\frac{\hat{\xi}_{t|t-1}\odot\eta_t}{1^T(\hat{\xi}_{t|t-1}\odot\eta_t)}, \tag{12}$$

$$\hat{\xi}_{t+1|t}=\boldsymbol{P}^T\hat{\xi}_{t|t}, \tag{13}$$

where $\hat{\xi}_{t|t}=P[s_t=j\mid Y_t;\theta]$ and $\hat{\xi}_{t+1|t}=P\left[s_{t+1}=j\middle|Y_t;\theta\right]$, the Hadamard's multiplication denoted by $\odot$ means the multiplication coordinate by coordinate. The vector of copulas' density is denoted by η_t,

$$\eta_t=\begin{bmatrix}c^{(1)}\left(F_1(y_{1t};\delta_1),F_2\left(y_{2t};\delta_2\right);\theta_1\right)\\ c^{(2)}\left(F_1(y_{1t};\delta_1),F_2\left(y_{2t};\delta_2\right);\theta_2\right)\end{bmatrix}.$$

The log likelihood function defined by (10) for the observed data can be written as:

$$L_c\left(\boldsymbol{Y};\delta,\theta\right)=\sum_{t=1}^{T}\ln\left(1^T\left(\hat{\xi}_{t|t-1}\odot\eta_t\right)\right),$$

where the initial value $\hat{\xi}_{1|0}$ is the limit probability vector:

$$\hat{\xi}_{1|0}=\begin{bmatrix}\dfrac{1-p_{22}}{2-p_{11}-p_{22}}\\ \dfrac{1-p_{11}}{2-p_{11}-p_{22}}\end{bmatrix}.$$

In the next section, we present estimation results for the five analyzed stocks.

6. Estimation results

The marginal distributions can be modelled by various methods. In this article a simple ARMA model was used. It may not seem to be the most efficient model. However, in respect to the investigating dependence, more advanced models of the margins lead to similar results. For all five series of realized volatility and the five series of daily trading volume an ARMA(2,1) model seems to be the proper one. In every set of estimation results, all parameters are significant. This suggests that the model is not overparametrized. Including more parameters does not improve results. The Table 5 and 6 present the estimated parameters for the five realized volatility series and the five daily trading volume series:

Table 5

Estimated parameters of an ARMA(2,1) model for realized volatility time series

	Andritz	Erste	OMV	TKA	Voest
ar1	1.1064	1.2801	1.2036	1.1155	1.1012
ar2	–0.1235	–0.2869	–0.2153	–0.1254	–0.1176
ma1	–0.7868	–0.8232	–0.8300	–0.8213	–0.7799
const	–7.7653	–7.8781	–8.0073	–8.1035	–7.6613

Table 6

Estimated parameters of an ARMA(2,1) model for daily volume time series

	Andritz	Erste	OMV	TKA	Voest
ar1	1.2213	1.2856	1.2802	1.2660	1.2930
ar2	–0.2417	–0.2932	–0.2968	–0.2689	–0.2965
ma1	–0.8644	–0.7937	–0.8716	–0.9505	–0.8568
const	11.9068	14.4653	13.3779	13.8284	13.6951

Using the estimation method presented in the previous section, the log likelihood function has been maximized. The Table 7 presents results of the estimation:

Table 7

Estimated parameters of switching copula model

	Copula(1)	p_{11}	p_{22}	θ	δ	ρ
Andritz	BB7	0.9963	0.8566	1.2007	0.1964	–0.4096
Erste	BB4	0.9831	0.8206	0.2380	0.4011	–0.2799
OMV	BB4	0.9705	0.9147	0.0628	0.1617	–0.2644
TKA	BB7	0.9926	0.7546	1.1398	0.1928	–0.2964
Voest	BB4	0.9879	0.7628	0.1581	0.4206	–0.2457

In table 7 the values p_{11} and p_{22} denote suitable transition probabilities, θ and δ stand for parameters of an asymmetric copula and ρ denotes the parameter of the Gaussian copula.

Applying formulas for lower and upper tail dependencies contained in table 3, it is possible for us to calculate tail dependencies between investigated pairs of variables. Clearly, both tail dependencies are equal to zero in the second regime. The Table 8 presents lower and upper tail dependencies in the first regime.

Table 8

Tail dependencies between realized volatility and daily volume in the first regime

	Andritz	**Erste**	**OMV**	**TKA**	**Voest**
λ_L	0.0293	0.0803	0.0002	0.0274	0.0236
λ_U	0.2188	0.1776	0.0137	0.1630	0.1924

In table 7, we see that a dependence in the upper tail is much stronger than in the lower tail for all the five of stocks. Both tail dependencies of OMV highly differ from tail dependencies of other stocks, in fact they are smaller. Furthermore, the lower tail dependence of realized volatility and daily volume of Erste series is noticeably higher than lower tail dependencies of other stocks.

The backward recursion provides us with the probability of being at the particular regime at the time t. It is given by:

$$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} \odot \{\boldsymbol{P}^T [\hat{\xi}_{t+1|T} (\div) \hat{\xi}_{t+1|t}]\},$$

where $\hat{\xi}_{t|t}$ and $\hat{\xi}_{t+1|t}$ are defined by (12) and (13), $\odot$ and (÷) denote multiplying and dividing coordinate by coordinate, respectively. From the practical point of view, the most relevant is vector $\hat{\xi}_{T|T}$. This vector is a probability vector of being at the particular regime at the time *T*. The Figure 1–5 presents smoothed probabilities of being at the first regime for five investigated stocks:

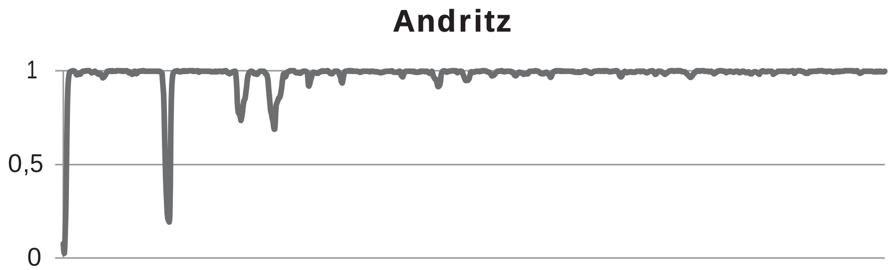

Figure 1. Estimated probabilities of being at the first regime for stock Andritz

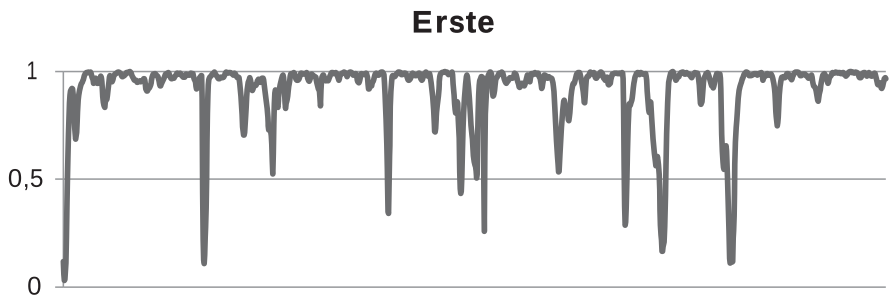

Figure 2. Estimated probabilities of being at the first regime for stock Erste

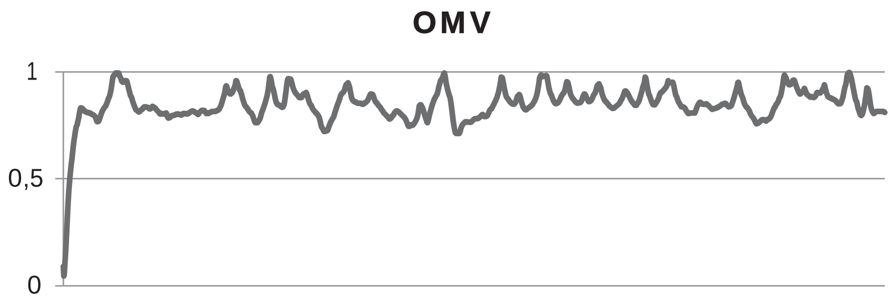

Figure 3. Estimated probabilities of being at the first regime for stock OMV

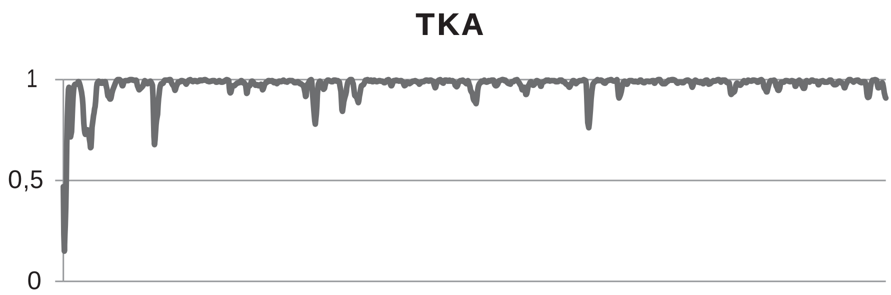

Figure 4. Estimated probabilities of being at the first regime for stock TKA

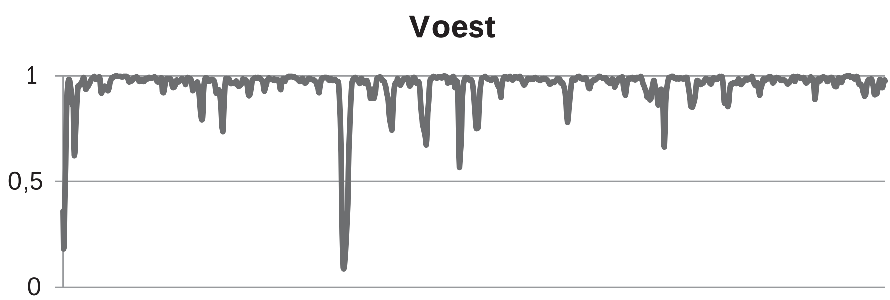

Figure 5. Estimated probabilities of being at the first regime for stock Voest

7. Concluding remarks

The knowledge of interdependencies between realized volatility and trading volume and their changes over the time may support investment decisions. The copula based regime switching models are flexible tools for modelling of the changes over the time period of the structure of interdependencies between return volatility and trading volume. The estimation of the model parameters allows researchers to compute the mean time of remaining the financial variable (e.g. equity price or trading volume) in a certain state and time of coming back to the previous state. The computations by means of copula based regime switching models delivered results concerning the interdependencies between realized return volatility and trading volume of selected companies listed in ATX.

A copula in the first regime was chosen as asymmetric copula with positive lower and upper tail dependencies. Conversely, Gaussian copula in the second regime is symmetric copula and variables linked with such copula are tail independent. For all analyzed stocks the probability of being at the first regime appeared to be vitally greater than being at the second regime. This result suggests that there is considerable dependence between realized volatility and daily volume in extreme values.

One can notice that a dependence in the upper tail is much more stronger than in the lower tail for all the five stocks in the first regime. Both tail dependencies of OMV were essentially smaller than the tail dependencies of other stocks.

In addition, the lower tail dependence of realized volatility and daily trading volume of Erste is significantly higher than in the case of other stocks under study. Since OMV and Erste are of similar capitalization the results suggest that the links between realized volatility and trading volume do not probably depend on the size of company but on the branch where a company is active.

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