Managerial Economics
2021, Vol. 22, No. 1, pp. 23–34
https://doi.org/10.7494/manage.2021.22.1.23

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An example to illustrate several aspects of optimization theory in Managerial Economics

1. Introduction

Optimization theory plays a significant role in Managerial Economics. Chapter 8 of Peterson and Lewis (1999) provides a lucid exposition of linear programming, followed by Mote and Madhavan (2016), where in chapters 5 and 22, there is a comprehensive and very informed treatment of the same topic and further discussions on integer programming and decision making under uncertainty. Neither of the two books discuss dynamic programming explicitly although simple integer programming problems can be solved easily by dynamic programming. Using dynamic programming for such integer programming leads to the representation of the problem by decision trees which are discussed in chapter 16 of Mote and Madhavan (2016), in the context of decision analysis.

As in chapter 5 of Mote and Madhavan (2016), where a single example is used to discuss almost all aspects of linear programming, it would be good to have a single example that illustrates all aspects of linear, integer and dynamic programming, including such concepts such as value of perfect and imperfect information. That is precisely what we do here.

The purpose of this paper is similar to Shenoy (1998), which is a seminal contribution to decision analysis from a purely pedagogic point of view. If in game theory or in a game tree a player whose turn it is to make a move, does not know the move that was chosen by the former's immediate predecessor, nor can the player whose turn it is to move identify its present position, then such a player is said to be located at an "information set". Exactly the same dilemma is faced by a decision maker whose move is preceded or followed by "chance". In either case, the pay-off of the player or the decision maker we are concerned

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with depends on its move as well as the unknown move of the other player or chance. In Shenoy (1998), the name “information set” is incorporated into decision trees at such nodes where owing to a move by chance, the decision maker is unaware of its exact location. Shenoy (1998) goes on to provide a solution for such a game tree under probabilistic uncertainty (risk) applied to a problem related to drilling of oil. As is well known, the consequences of drilling in a “suspected” oil field are uncertain.

Our example allows for the availability of “additional information” (as for instance a preliminary geological survey to update the existing information regarding the availability of oil) at a price. All of the above and this embedded in a linear programming problem is to the best of our knowledge a novelty for a learner of decision analysis, if not for practitioners as well.

2. The mathematical background

Here we provide the general model in the context of which our discussion takes place.

Given positive integers m, n and M a subset of {1,…, n}, the standard form of the general problem we are concerned with is the following

$$\text{Maximize } \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, \ldots, m,$$

$x_j \geq 0$ for j = 1, ..., n, $x_j \in \mathbb{N} \cup \{0\}$ for $j \in M$, where $\mathbb{N}$ is the set of natural numbers.

If M = ϕ, then the above is a linear programming (LP) problem, which from the perspective of managerial economics is covered extremely well in both Petersen and Lewis (1999) and Mote and Madhavan (2016). Technical details for such problems are available in Lahiri (2021).

If M = $\mathbb{N}$, then we have an integer programming problem. If in addition we require some variable $x_j \in \{0,1\}$, then we simply add the inequality $x_j \leq 1$ to the above system, unless it is already there.

Sometimes there may be probabilistic uncertainty about certain parameters of the above problem. In such a situation it may be possible to obtain information about the uncertain parameters which leads to an improved value of the objective. The difference in the value of the objective function- after and prior to the availability of information- is called the value of information. This value may depend on whether the information about the uncertain parameters is perfect or imperfect. The important thing to note about information is that it should be available when required.

The example in the next section gives us a peep into the issues discussed above.

3. Numerical example

The following numerical example can be used for instructional purposes to explain the issues mentioned above.

Maximize $x_1 + r(x_3)x_2 - \frac{3}{4}x_3$ where $\Pr.\{r(x_3) = \frac{3}{2} \mid x_3 = 1\} = \frac{7}{8}$, $\Pr.\{r(x_3) = = \frac{1}{2} \mid x_3 = 1\} = \frac{1}{8}$, $\Pr.\{r(x_3) = \frac{3}{2} \mid x_3 = 0\} = \frac{1}{8}$, $\Pr.\{r(x_3) = \frac{1}{2} \mid x_3 = 0\} = \frac{7}{8}$.

To be precise r(0) and r(1) are two independent random variables.

If in addition we require x_1 and x_2 to be non-negative integers, then we have an integer programming problem and such simple integer programming problems can be solved using decision trees, i.e. dynamic programming.

One can also discuss value of perfect and imperfect information, so that if the person providing information is known to be correct with probability $p(x_3)$, then $r(x_3)$ is the predicted value with probability $p(x_3)$ and the other value with probability $1-p(x_3)$. We could generalize this further by letting $\rho(x_3)$ denote the predicted value of the co-efficient of x_2 for a given value of x_3 and considering $p(r(x_3) = \rho(x_3) \mid \rho(x_3) = \alpha)$ = Probability of the event $[r(x_3) = \rho(x_3)]$ conditional on the event $[\rho(x_3) = \alpha]$ and $1- p(r(x_3) = \rho(x_3) \mid \rho(x_3) = \alpha)$ = Probability of the event $[r(x_3) \in \{\frac{1}{2}, \frac{3}{2}\} \backslash \{\rho(x_3)\}]$ conditional on the event $[\rho(x_3) = \alpha]$, for $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$. However, that would just be complicating the calculations and is left as an exercise for the interested reader.

In any case we would require to obtain $\Pr.[\rho(x_3) = \alpha]$ = Probability of the event $[\rho(x_3) = \alpha]$, which can be done using Baye's rule (please see Appendix for details).

In our case, $\Pr.[\rho(x_3) = \alpha] = \dfrac{\text{Probability of the event}\left[r(x_3) = \alpha\right] - (1 - p(x_3))}{2p(x_3) - 1}$.

Hence $\Pr.[\rho(0) = \frac{1}{2}] = \dfrac{\frac{7}{8} - (1 - p(0))}{2p(0) - 1} = \dfrac{p(0) - \frac{1}{8}}{2p(0) - 1}$, $\Pr.[\rho(0) = \frac{3}{2}] = \dfrac{\frac{1}{8} - (1 - p(0))}{2p(0) - 1} = \dfrac{p(0) - \frac{7}{8}}{2p(0) - 1}$,

$\Pr.[\rho(1) = \frac{1}{2}] = \dfrac{\frac{1}{8} - (1 - p(1))}{2p(1) - 1} = \dfrac{p(1) - \frac{7}{8}}{2p(1) - 1}$, $\Pr.[\rho(1) = \frac{3}{2}] = \dfrac{\frac{7}{8} - (1 - p(1))}{2p(1) - 1} = \dfrac{p(1) - \frac{1}{8}}{2p(1) - 1}$.

In order to determine the value of x_3 it is necessary for the DM to have information about r(0) and r(1) right at the beginning of the decision making process.

4. Solution of the numerical example for an LP and the value of perfect information

In the absence of any information we compare the value of the optimal solutions for $x_3 = 0$ and $x_3 = 1$ using $r(0) = \frac{3}{2}$ with probability $\frac{1}{8}$, $r(0) = \frac{1}{2}$ with probability $\frac{7}{8}$ and $r(1) = \frac{3}{2}$ with probability $\frac{7}{8}$, $r(1) = \frac{1}{2}$ with probability $\frac{1}{8}$ and = $\frac{11}{8}$ and choose the solution which gives the higher optimal value.

Hence we solve

Maximize $x_1 + \frac{3}{2}x_2$
s.t. $2x_1 + x_2 \leq 4$
$x_1 + 2x_2 \leq 4$,
$x_1 \geq 0$, $x_2 \geq 0$,

and

Maximize $x_1 + \frac{1}{2}x_2$
s.t. $2x_1 + x_2 \leq 4$
$x_1 + 2x_2 \leq 4$,
$x_1 \geq 0$, $x_2 \geq 0$.

Without any integer constraints, we know from LP theory that if an optimal solution exists then there must be one at one of the four corner points $\{(0,0), (0,2), (2,0), (\frac{4}{3}, \frac{4}{3})\}$. Since the set of values of the objective function for both problems corresponding to the set of feasible points is bounded above, it is known (a proof is available in Lahiri (2020) that optimal solutions exist for both problems.

The optimal solution for the first LP problem i.e. the one with co-efficient of x_2 being $\frac{3}{2}$ is $(\frac{4}{3}, \frac{4}{3})$ with optimal value being $\frac{10}{3}$.

The set of optimal solution for the first LP problem i.e. the one with co-efficient of x_2 being $\frac{1}{2}$ is the closed interval joining the end points $(\frac{4}{3}, \frac{4}{3})$ and (2,0), with optimal value being 2.

Hence the expected optimal value after choosing $x_3 = 0$ is $\frac{7}{8}\times\frac{1}{2} + \frac{1}{8}\times\frac{3}{2} = \frac{10}{16} = \frac{5}{8}$ and the optimal value after choosing $x_3 = 0$ is $\frac{7}{8}\times\frac{3}{2} + \frac{1}{8}\times\frac{1}{2} - \frac{3}{4} = \frac{22}{16} - \frac{12}{16} = \frac{10}{16} = \frac{5}{8}$.

Hence the DM is indifferent between choosing $x_3 = 0$ and $x_3 = 1$, and having chosen x_3 waits for the realized value of $r(x_3)$ to decide what the optimal values of x_1 and x_2 should be.

The expected optimal value without any information is $\frac{5}{8}$.

If perfect information is available, then there are four possibilities for the pairs of predicted values of r: $(r(0),r(1)) = (\frac{3}{2}, \frac{3}{2})$, $(r(0),r(1)) = (\frac{1}{2}, \frac{1}{2})$, $(r(0),r(1)) = (\frac{3}{2}, \frac{1}{2})$ and $(r(0),r(1)) = (\frac{1}{2}, \frac{3}{2})$.

From the above calculations we know that the optimal value pairs corresponding to (r(0), r(1)) =

(a) $(\frac{3}{2}, \frac{3}{2})$ is $(\frac{10}{3}, \frac{10}{3} - \frac{3}{4})$ with the optimal solution in both situations being $(x_1, x_2) = (\frac{4}{3}, \frac{4}{3})$;

(b) $(\frac{3}{2}, \frac{1}{2})$ is $(\frac{10}{3}, 2 - \frac{3}{4}) = (\frac{10}{3}, \frac{5}{4})$ with the optimal solution for $x_3 = 0$ being $(x_1, x_2) = (\frac{4}{3}, \frac{4}{3})$ and the set of optimal solutions for $x_3 = 1$ being ordered pairs (x_1, x_2) in the closed interval joining the end points $(\frac{4}{3}, \frac{4}{3})$ and (2,0);

(c) $(\frac{1}{2}, \frac{3}{2}) = (2, \frac{10}{3} - \frac{3}{4}) = (2, \frac{91}{12})$ with the set of optimal solutions for $x_3 = 0$ being ordered pairs (x_1, x_2) in the closed interval joining the end points $(\frac{4}{3}, \frac{4}{3})$ and (2,0) and the optimal value for $x_3 = 1$ being $(x_1, x_2) = (\frac{4}{3}, \frac{4}{3})$;

(d) $(\frac{1}{2}, \frac{1}{2})$ is $(2, 2 - \frac{3}{4}) = (2, \frac{5}{4})$ with the set of optimal solution for both $x_3 = 0$ and $x_3 = 1$ being ordered pairs (x_1, x_2) in the closed interval joining the end points $(\frac{4}{3}, \frac{4}{3})$ and (2, 0).

If the predicted value of $r(0) = \frac{3}{2}$, then the optimal choice is $(\frac{4}{3}, \frac{4}{3}, 0)$ with an optimal value of $3\frac{1}{3}$. The probability of such a prediction is $\frac{1}{8}$.

If the prediction is $(r(0),r(1)) = (\frac{1}{2}, \frac{1}{2})$, then the optimal choice is any point in the closed interval with end points $(\frac{4}{3}, \frac{4}{3}, 0)$ and (2, 0, 0) with an optimal value of 2. The probability of such a prediction is $\frac{7}{64}$.

If the prediction is $(r(0),r(1)) = (\frac{1}{2}, \frac{3}{2})$, then the optimal choice is $(\frac{4}{3}, \frac{4}{3}, 1)$ with an optimal value of $2\frac{17}{12}$. The probability of such a prediction is $\frac{49}{64}$.

Hence with perfect information, the optimal expected value of the objective function is $3\frac{1}{3} \times \frac{1}{8} + 2 \times \frac{7}{64} + 2\frac{7}{12} \times \frac{49}{64} = 2\frac{673}{2304}$.

The optimal value of the objective function without any information is $\frac{5}{8}$ and the optimal value of the objective function with perfect information is $2\frac{673}{2304}$.

Hence the value of perfect information is $2\frac{673}{2304} - \frac{5}{8}$ is $2 - \frac{17}{144} = 1\frac{673}{1440} > 0$.

5. LP and the value of imperfect information

Suppose for $x_3 \in \{0, 1\}$, there is a probability $p(x_3) \in [0,1]$ such that the predicted value of $r(x_3)$ is correct. Recall that $\rho(x_3)$ denotes the predicted value and $r(x_3)$ denotes the realized value for $x_3 \in \{0, 1\}$. Thus, $p(x_3)$ is the probability of the event $\{r(x_3) = \rho(x_3)\}$. In the previous section we were assuming $p(x_3) = 1$ for $x_3 \in \{0,1\}$. In this section, we relax this assumption. Thus, for $x_3 \in \{0,1\}$, $r(x_3) = \rho(x_3)$ with probability $p(x_3)$ and $r(x_3) \in \{\frac{1}{2}, \frac{3}{2}\} \backslash \{\rho(x_3)\}$, with probability $1 - p(x_3)$.

If $\rho(0) = \frac{1}{2}$, then with probability p(0), $r(0) = \frac{1}{2}$ with the optimal value of the corresponding problem being 2 and with probability $1 - p(0)$, $r(x_3) = \frac{3}{2}$ with the optimal value of the corresponding problem being $3\frac{1}{3}$. Hence if $\rho(0) = \frac{1}{2}$, the expected optimal value of the DM is $2p(0) + 3\frac{1}{3}(1 - p(0)) = 3\frac{1}{3} - 1\frac{1}{3}\ p(0)$.

Similarly if $\rho(0) = \frac{3}{2}$, the expected optimal value of the DM is $3\frac{1}{3}\ p(0) +$ $+ 2(1 - p(0)) = 2 + 1\frac{1}{3}\ p(0)$.

If $\rho(1) = \frac{1}{2}$, then the expected optimal value of the DM is $2p(1) + 3\frac{1}{3}(1 - p(1))$ $- \frac{3}{4} = 3\frac{1}{3} - 1\frac{1}{3}\ p(1) - \frac{3}{4} = 2\frac{5}{9} - 1\frac{1}{3}\ p(1)$.

If $\rho(1) = \frac{3}{2}$, the expected optimal value of the DM is $3\frac{1}{3}\ p(1) + 2(1 - p(1))$ $- \frac{3}{4} = \ = 2 - \frac{3}{4} + 1\frac{1}{3}\ p(1) = 1\frac{1}{4} + 1\frac{1}{3}\ p(1)$.

As before there are four possibilities for the pairs of predicted values of r: $(\rho(0), \rho(1)) = (\frac{3}{2}, \frac{3}{2})$, $(\rho(0), \rho(1)) = (\frac{1}{2}, \frac{1}{2})$, $(\rho(0), \rho(1)) = (\frac{3}{2}, \frac{1}{2})$ and $(\rho(0), \rho(1)) = (\frac{1}{2}, \frac{3}{2})$.

If $(\rho(0), \rho(1)) = (\frac{3}{2}, \frac{3}{2})$, then the DM will choose $x_3 = 0$ or 1, depending upon whether $2 + 1\frac{1}{3}\ p(0)$ is greater than or equal to $1\frac{1}{4} + 1\frac{1}{3}\ p(1)$ or the other way around. Hence the DM's expected optimal value is $\max\{2 + 1\frac{1}{3}\ p(0), 1\frac{1}{4} + 1\frac{1}{3}\ p(1)\}$. The probability of the prediction being $(\rho(0),\rho(1)) = (\frac{3}{2}, \frac{3}{2})$ is $\left(\frac{p(0)-\frac{7}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{1}{8}}{2p(1)-1}\right)$.

If $(\rho(0), \rho(1)) = (\frac{1}{2}, \frac{1}{2})$ the DM's expected optimal value is $\max\{3\frac{1}{3} - 1\frac{1}{3}\ p(0), 2\frac{5}{9} - 1\frac{1}{3}\ p(1)\}$. The probability of the prediction being $(\rho(0), \rho(1)) = (\frac{1}{2}, \frac{1}{2})$ is $\left(\frac{p(0)-\frac{1}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{7}{8}}{2p(1)-1}\right)$.

If $(\rho(0), \rho(1)) = (\frac{3}{2}, \frac{1}{2})$ the DM's expected optimal value is $\max\{2 + 1\frac{1}{3}\ p(0), 2\frac{5}{9} - 1\frac{1}{3}\ p(1)\}$. The probability of the prediction being $(\rho(0), \rho(1)) = (\frac{3}{2}, \frac{1}{2})$ is $\left(\frac{p(0)-\frac{7}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{7}{8}}{2p(1)-1}\right)$.

If $(\rho(0), \rho(1)) = (\frac{1}{2}, \frac{3}{2})$ the DM's expected optimal value is $\max\{3\frac{1}{3} - 1\frac{1}{3}\ p(0), 1\frac{1}{4} + 1\frac{1}{3}\ p(1)\}$. The probability of the prediction being $(\rho(0), \rho(1)) = (\frac{1}{2}, \frac{3}{2})$ is $\left(\frac{p(0)-\frac{1}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{1}{8}}{2p(1)-1}\right)$.

Hence with imperfect information, the optimal expected value of the objective function is $[\left(\frac{p(0)-\frac{7}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{1}{8}}{2p(1)-1}\right)][\max\{2 + 1\frac{1}{3}\ p(0), 1\frac{1}{4} + 1\frac{1}{3}\ p(1)\}] + [\left(\frac{p(0)-\frac{1}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{7}{8}}{2p(1)-1}\right)]$ $[\max\{3\frac{1}{3} - 1\frac{1}{3}\ p(0), 2\frac{5}{9} - 1\frac{1}{3}\ p(1)\}] + [\left(\frac{p(0)-\frac{7}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{7}{8}}{2p(1)-1}\right)][\max\{2 + 1\frac{1}{3}\ p(0),$ $2\frac{5}{9} - 1\frac{1}{3}\ p(1)\}] + \left[\left(\frac{p(0)-\frac{1}{8}}{2p(0)-1}\right)\left(\frac{p(1)-\frac{1}{8}}{2p(1)-1}\right)\right]\left[\max\left\{3\frac{1}{3} - 1\frac{1}{3}p(0), 1\frac{1}{4} + 1\frac{1}{3}p(1)\right\}\right]$.

Without any information the optimal expected value of the objective function is $\frac{5}{8}$.

The value of imperfect information is the difference between the optimal expected value of the objective function with imperfect information and $\frac{5}{8}$.

If it is positive, then the value of imperfect information is the maximum the DM is willing to pay for obtaining imperfect information.

If p(0) = p(1) = 1, then the above sum reduces to $\frac{7}{64}[3\frac{1}{3} + 2] + \frac{1}{64} \times 3\frac{1}{3} + \frac{49}{64} \times$ $\times 2\frac{7}{12} = \frac{7}{64} \times 5\frac{1}{3} + \frac{1}{64} \times 3\frac{1}{3} + \frac{49}{64} \times 2\frac{7}{12} = 2\frac{673}{2304}$.

If p(0) and p(1) are sufficiently close to 1, then the value of imperfect information is likely to be positive.

6. The integer programming version of the above problem

The integer programming version of the above problem is the following

Maximize $x_1 + r(x_3)x_2 - \frac{3}{4}x_3$
s.t. $2x_1 + x_2 \le 4$
$x_1 + 2x_2 \le 4$
$x_3 \le 1$,
$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$,
where Pr.$\{r(x_3) = \frac{3}{2} \mid x_3 = 1\} = \frac{7}{8}$, Pr.$\{r(x_3) = \frac{1}{2} \mid x_3 = 1\} = \frac{1}{8}$,
Pr.$\{r(x_3) = \frac{3}{2} \mid x_3 = 0\} = \frac{1}{8}$, Pr.$\{r(x_3) = \frac{1}{2} \mid x_3 = 0\} = \frac{7}{8}$.

Once again, r(0) and r(1) are two independent random variables.

The analysis differs from the above only in the computational strategies of the following two integer linear programming problems:

Maximize $x_1 + \frac{3}{2}x_2$
s.t. $2x_1 + x_2 \le 4$
$x_1 + 2x_2 \le 4$
$x_1, x_2 \in \mathbb{N} \cup \{0\}$,

and

Maximize $x_1 + \frac{1}{2}x_2$
s.t. $2x_1 + x_2 \le 4$
$x_1 + 2x_2 \le 4$
$x_1, x_2 \in \mathbb{N} \cup \{0\}$.

In this situation it is easy to observe that for both problems the set of feasible solutions is {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)}.

The optimal solution for the first problem is at (0, 2) with the optimal value being 3 and the optimal solution for the second problem is at (2, 0) with the optimal value being 2. However, the interesting point to note is that solving the original problem using a decision tree can be quite instructive about several aspects of managerial decision analysis.

At the root of the tree, which is a node, the DM chooses an action/move i.e. an edge of the tree, from the two edges $x_3 = 0$, $x_3 = 1$. A note where the DM, has a move, called a decision node, is usually denoted by a square.

At the next pair of nodes, regardless of what the choice at the root of the tree, it is the turn for chance to make a move. A note where chance has a move, called a chance node, is usually denoted by a circle.

If $x_3 = 0$, then chance chooses the edge $[r(0) = \frac{1}{2}]$ with probability $\frac{7}{8}$ and $[r(0) = \frac{3}{2}]$ with probability $\frac{1}{8}$. Let the resulting nodes be denoted $IP^1(0)$ and $IP^2(0)$. These are decision nodes with the states variables $(b_1, b_2) = (4, 4)$ and value $V_2 = 0$ inscribed within it. From this node the DM, is required to choose one of three possible edges corresponding to the three values of x_2: $x_2 = 0$, $x_2 = 1$, $x_2 = 2$.

If $x_3 = 1$, then chance chooses the edge $[r(1) = \frac{1}{2}]$ with probability $\frac{1}{8}$ and $[r(1) = \frac{3}{2}]$ with probability $\frac{7}{8}$. Let the resulting nodes be denoted $IP^1(1)$ and $IP^2(0)$. These are decision nodes with the state variables $(b_1, b_2) = (4, 4)$ and value $V_2 = -\frac{3}{4}$ inscribed within it. From this node the DM, is required to choose one of three possible edges corresponding to the three values of x_2: $x_2 = 0$, $x_2 = 1$, $x_2 = 2$.

If $x_3 = 0$, then for the chosen values of r(0) and x_2 we arrive at a decision node with the state variable $(b_1, b_2) = (4 - x_2, 4 - 2x_2)$ and value $V_1 = r(0)x_2$ inscribed within it.

If $x_3 = 1$, then for the chosen values of r(1) and x_2 we arrive at a decision node with the state variable $(b_1, b_2) = (4 - x_2, 4 - 2x_2)$ and value $V_1 = -\frac{3}{4} + r(1)$ x_2 inscribed within it.

At the decision node with $(b_1, b_2) = (4 - x_2, 4 - 2x_2)$ and value $V_1 = r(0)x_2$ inscribed within it, the possible values of x_1 are all non-negative integers less than or equal to $\min\{\frac{4-x_2}{2}, 4 - 2x_2\}$, with an edge corresponding to each such non-negative integer. At the end of such an edge is a terminal node of the tree with $V_0 = V_1$ + the value of x_1 along the chosen edge = $r(0)x_2$ + the value of x_1 along the chosen edge.

At the decision node with $(b_1, b_2) = (4 - x_2, 4 - 2x_2)$ and value $V_1 = -\frac{3}{4} + r(1)x_2$ inscribed within it, the possible values of x_1 are all non-negative integers less than or equal to $\min\{\frac{4-x_2}{2}, 4 - 2x_2\}$, with an edge corresponding to each such non-negative integer. At the end of such an edge is a terminal node of the tree with $V_0 = V_1$ + the value of x_1 along the chosen edge $= -\frac{3}{4} + r(1)x_2$ + the value of x_1 along the chosen edge.

Since the optimization problem is a maximization problem with the co-efficient of x_1 being positive at all decision nodes at the last stage of the decision tree the chosen value of x_1 will be $\min\{\frac{4-x_2}{2}, 4 - 2x_2\}$.

Thus if $x_3 = 0$, then for the value of r(0) chosen by chance the chosen value of x_2 must be a maximizer of $r(0)x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\}$ which is $\max\{\frac{1}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0, 1, 2\}\}$ with probability $\frac{7}{8}$ and $\max\{\frac{3}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0, 1, 2\}\}$ with probability $\frac{1}{8}$.

Thus, the optimal expected value resulting from choosing $x_3 = 0$ is $\frac{7}{8} \times [\max\{\frac{1}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0, 1, 2\}\}] + \frac{1}{8} \times [\max\{\frac{3}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0,1,2\}\}]$. Let us call this $EV(x_3 = 0)$.

Similarly if $x_3 = 1$, then for the value of r(1) chosen by chance the chosen value of x_2 must be a maximizer of $-\frac{3}{4} + r(1)x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\}$ which is $\max\{-\frac{3}{4} + \frac{1}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0,1,2\}\}$ with probability $\frac{1}{8}$ and $\max\{-\frac{3}{4} + \frac{3}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0,1,2\}\}$ with probability $\frac{7}{8}$.

Thus, the optimal expected value resulting from choosing $x_3 = 1$ is $\frac{7}{8} \times [\max\{-\frac{3}{4} + \frac{1}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0, 1, 2\}\}] + \frac{1}{8} \times [\max\{\frac{1}{2}x_2 + \min\{\frac{4-x_2}{2}, 4 - 2x_2\} | x_2 \in \{0, 1, 2\}\}]$. Let us call this $EV(x_3 = 1)$.

An optimal solution for x_3 is equal to 0 if and only if $EV(x_3 = 0) \geq EV(x_3 = 1)$. Otherwise, the optimal value of x_3 is equal to 1.

In our problem $EV(x_3 = 0) = \frac{7}{8} \times 2 + \frac{1}{8} \times 4 = \frac{18}{8} = 2\frac{1}{4}$ and $EV(x_3 = 1) = \frac{7}{8} \times 3 + \frac{1}{8} \times 2 - \frac{3}{4} = \frac{17}{8} = 2\frac{1}{8}$.

Thus $EV(x_3 = 0) > EV(x_3 = 1)$ and hence the optimal choice is $x_3 = 0$.

Contrast this result with the one we obtained for LP without any information.

The result for the cases with perfect and imperfect information for an IP are obtained in an analogous manner for that of an LP.

Acknowledgment

I would like to gratefully and respectfully acknowledge the encouraging comments received from Professor Prasanta Pattanaik ("interesting in its own right") and Professor Katta G Murty (about the exact fit of the example with the intended audience) concerning this document. Needless to say, that I simultaneously thank and apologise to those students who had to suffer silently the several *visions and revisions of the above problem. I would also like to thank Supriya Koul for WhatsApp discussions about the matter reported in the Appendix. Finally, I would like to put on record my deep gratitude to an anonymous referee and Editors of this suggestion for valuable suggestions towards improvement of this paper. However, none other than the author is responsible for residual errors.*

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Appendix

r and ρ are two random variables defined on a set A consisting of two elements a generic element of which is denoted by α.

Probability $[r = \alpha]$ and Probability $[r = \alpha|\ \rho = \beta] = p(r = \alpha|\ \rho = \beta)$ known for all $\alpha, \beta \in A$. We need to find Probability $[\rho = \alpha]$ for all $\alpha \in A$.

$p(r = \alpha|\rho = \alpha) \times \Pr.[\rho = \alpha]$ = Probability of the event $[r = \alpha\ \&\ \rho = \alpha)]$.

$p(r = \alpha|\rho \in A\backslash\{\alpha\}) \times \Pr.[\rho \in A\backslash\{\alpha\}]$ = Probability of the event $[r = \alpha\ \&\ A\backslash\{\alpha\}]$.

Adding the two equations we get $p(r = \alpha|\rho = \alpha) \times \Pr.[\rho = \alpha] + p(r = \alpha|\rho \in A\backslash\{\alpha\})$ $(1 - \Pr.[\rho = \alpha])$ = Probability of the event $[r = \alpha]$.

Thus, $\Pr.[\rho = \alpha] =$

$$\frac{\text{Probability of the event}\left[r=\alpha\right]-p\left(r=\alpha\,|\,\rho\in A\setminus\{\alpha\}\right)}{p\left(r=\alpha\,|\,\rho=\alpha\right)-p\left(r=\alpha\,|\,\rho\in A\setminus\{\alpha\}\right)}=$$

$$=\frac{p\left(r=\alpha\,|\,\rho\in A\setminus\{\alpha\}\right)-\text{Probability of the event}\left[r=\alpha\right]}{p\left(r=\alpha\,|\,\rho\in A\setminus\{\alpha\}\right)-p\left(r=\alpha\,|\,\rho=\alpha\right)}$$

Note: If Probability of the event $[r = \alpha] > p(r = \alpha|\rho = \alpha)$, then if $1 > \Pr.[\rho = \alpha] > 0$, $\Pr.[\rho = \alpha] \times$ Probability of the event $[r = \alpha] > \Pr.[\rho = \alpha] \times$ $\times\ p(r = \alpha|\rho = \alpha)$.

Since Probability of the event $[r = \alpha]$ = Pr.$[\rho = \alpha] \times$ Probability of the event $[r = \alpha]$ + (1 – Pr.$[\rho = \alpha]) \times$ Probability of the event $[r = \alpha]$, it must be the case that (1 – Pr. $[\rho = \alpha]) \times$ Probability of the event $[r = \alpha] < p(r = \alpha | \rho \in A \setminus \{\alpha\}) \times (1 - \text{Pr.}[\rho = \alpha])$, and hence Probability of the event $[r = \alpha] < p(r = \alpha | \rho \in A \setminus \{\alpha\})$.

The converse is also true, as can be checked from the calculations above.

Summary

We provide a single example that illustrates all aspects of linear, integer and dynamic programming, including such concepts such as value of perfect and imperfect information. Such problems, though extremely plausible and realistic are hardly ever discussed in managerial economics.

JEL codes: A22, A23, C61, D01, M21

Keywords: *managerial economics, optimization, linear programming, decision making under risk, value of information*