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Consumer surplus and budget constrained preference maximization: A note

This paper is in honor of our (M.Stat(QE)-1980-81 batch) teacher of "Cost-Benefit Analysis" at ISI-Kolkata.

Introduction, motivation and discussion

As a teacher of microeconomics at a business school, one is periodically confronted with news items such as this: https://www.citylab.com/transportation/2016/09/uber-consumer-surplus/500135/.

If you read this article then you will see it makes non-trivial use of consumer surplus (= willingness to pay – what one pays) as a measure of consumer welfare. I would be honestly delighted if someone could produce a similar article addressed to business professionals and/or economists using equivalent variation (EV) and/or compensating variation (CV). Consumer surplus is a measure of welfare, whereas CV and EV are measures of change in welfare. Naturally, there are many statements using consumer surplus that cannot be phrased using CV and EV. As we all know, willingness to pay is conventionally measured by the area under the demand curve. Section 7.5 of Katzner (1970) contains a detailed discussion on consumer surplus. Consumer surplus along with the other welfare concepts mentioned above are defined and discussed in chapter 12 of Mandy (2017).

This leads to a certain amount of discomfort about the relationship between undergraduate microeconomics (or managerial economics) and graduate microeconomics. In reality, what is applied, and particularly for policy purposes, is undergraduate microeconomics or its managerial version, and not the microeconomics that is taught in graduate programs in economics. However, if there is a gap or

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inconsistency on a topic between the two, then one invariably tries to pass off the undergraduate level explanation as a simplification/approximation of the 'gospel truth' that is apparently conveyed in graduate level microeconomics. Thus, for instance, undergrad demand theory is passed off as utility maximization subject to budget constraint assuming utility functions are quasi-linear. Quasi-linear utility means utility of money held along with non-monetary consumption goods is equal to money held, plus utility derived from consumption goods. Assuming budget constrained quasi-linear utility functions renders undergraduate microeconomics horrendously inadequate, since then we cannot talk about normal, inferior or luxury goods. We cannot say that if income goes up then the demand curve will shift upwards (or downwards). Income change has no effect on the demand curve. There would be very little meaning left in undergraduate microeconomics. Even so, such an inadequate answer sounds credible, since the entire approach towards undergraduate microeconomics is based on 'hand waving'. There is no stand-alone rigorous approach to undergraduate microeconomics. Undergraduate students are told that if they pursue graduate economics then they will be told what real microeconomics is all about. And, what are they told if and when they enroll for graduate economics? They are told that the utility functions may be Cobb-Douglas – a favorite of Applied General Equilibrium Theory. Cobb-Douglas utility functions say that for every commodity there is a fixed fraction, so that regardless of what prices are a consumer's expenditure on the good is that fixed fraction of his disposable income. Hence if when your income is US$ 50,000 per month, the price of rice is US$ 1 per kilogram, and you are buying 5 kilograms per month, then for the same income, if for some reason price of rice shot up to US$ 4 per kilogram, you would be buying 1.25 kilograms for the entire month. Is such a form of microeconomics the best that is possible? I feel we can do better if we try to make the foundations of undergraduate microeconomics more rigorous and require a certain minimum amount of consistency between graduate level microeconomics and the microeconomic theory taught at the undergraduate level.

We share in an appendix of this paper a teaching note about an important issue in undergraduate and MBA level microeconomics, written for those who are interested in the logical foundations of what they are taught or what they are teaching. We are all for improvement, generalization and more rigor in the mathematics that is involved in the exposition. We leave that as a significant mission for further research. However, the economic theory reported in the appendix is not negotiable, since that is precisely what the paper is about.

In undergraduate consumer demand theory as well as in managerial economics, we teach that the consumer surplus is the area under the inverse demand curve up to the quantity consumed minus the expenditures. Implicitly what we mean is that the area under the inverse demand curve measures the Willingness To Pay

(WTP), so that when expenditure on the good is subtracted from WTP we have Consumer's Surplus (CS). But when is the area under the demand curve the WTP? The first major result in this teaching note says that *willingness to pay is the area under the demand curve if and only if consumers are surplus maximizers*. The 'mathematical jungle' which this and the ancillary results give rise to is purely because we show that this result holds not only for an individual consumer in a market, but also when we aggregate across commodities or individuals or both, so that it holds for the macroeconomic AS-AD (Aggregate Supply-Aggregate Demand) model as well. Yet why should that be a matter of any importance? The reason for its importance is that the inverse demand function depends not only on the quantity of the good that is consumed, but also on the prices of other goods and income. Hence, in general, WTP depends not only on the quantity of the good that is consumed, but prices of other goods and income. Furthermore, WTP **cannot** be the utility from the non-monetary goods consumed and consumer's surplus **is not** (any kind of) utility minus expenditure. If we assume that WTP does not depend on income then we can of course assume that WTP is the monetary worth of consuming non-monetary goods, i.e. the consumer is a budget constrained quasi-linear utility maximizer. This as we observed earlier trivializes consumer demand theory, contrary to the requirements of economic policy.

There is a common misconception that willingness to pay is a measure of satisfaction and further that it is interpersonally comparable as a measure of satisfaction. An example of this confusion or misunderstanding is available at the following the link: https://worldpolicy.org/2016/01/29/consumer-surplus-and-related-absurdities/

While willingness to pay may well qualify to be a monetary measure of the benefits that accrues to an individual or a group, it is certainly not a measure of satisfaction, particularly when it comes to interpersonal comparisons of satisfaction. Hence, the fact that Bill Gates may be willing to pay more for a slice of pizza than a poor and hungry person does not conflict with the equally convincing conjecture that Bill Gates gets less satisfaction from the slice of pizza than the latter does. It only means that Bill Gates, being a rich person, can and is willing to pay a huge amount of money and considerably more for a wee bit of satisfaction than what the poor man is willing to pay for a huge amount of satisfaction, provided satisfaction is numerically measurable. It does not conflict with the assumption of diminishing marginal utility of money, even in the extreme situation (such as the one assumed by the author of the article posted at the above link) where everyone has the same utility function for money. The same argument is true for an individual. On a hot summer day in the desert regions of western India, if a thirsty person is willing to pay more for the first pouch of cold potable water (sold in polythene bags) more than the second pouch, it does not mean nor is it required that the individual is indifferent between the consumption bundle consisting of one pouch of water and his income reduced by the price of a pouch of water and the consumption bundle consisting of two pouches of water and his income reduced by the expenses for two pouches of water. He could prefer the former consumption bundle to the latter consumption bundle or the other way round. Which consumption bundle the individual prefers between the two would depend on what the price of a pouch of potable water is. It may be reasonable to assume that more benefit (measured in monetary units) is usually preferred to less of it *ceteris paribus* by an individual or group of individuals. That, however, does not justify heroic conclusions about its relationship with more or less satisfaction. Cost benefit analysis, whether for an individual or a society, is a decision making procedure based on accounting and is thus concerned with economic welfare from a purely accounting perspective. Reckless cognitive or psychological interpretations of such and related concepts are totally unwarranted. Further, since we are not concerned with any psychological interpretation of willingness to pay, the kind of representation of surplus maximization by using a quasi-linear utility function, that several generations of economists have been exposed to from their undergraduate days, and is hopefully represented for the last time in chapters 3 and 4 of (Hayashi 2017), is not relevant for our analysis. The unfortunate misunderstanding of consumer surplus as a *ceteris paribus* concept (consequent to willingness to pay being defined that way in this paper) and the controversy surrounding it, is reflected in the discussions in pages 268–269 and the concluding section of chapter 12 of (Mandy 2017).

In Katzner (2008) it has been observed that if the individual consumer is a utility maximizer and if P' and P" are two price vectors, then the difference in the utility of the baskets of commodities that he/she demands at P' and P" is the sum, appropriately adjusted, of the areas under the relevant individual demand curves. That is, the sum of all the consumer surpluses (again appropriately adjusted) is a measure of the consumer's CHANGE in welfare as prices P' change to P" or visa versa. (Thus neither equivalent variation nor compensating variation is necessary to deal with welfare changes.) Under additional assumptions, this proposition is generalized to many consumers on p. 510. No assumption of surplus maximization is required. It is probably worth mentioning that similar results are available in (Takayama 1982, 1984) and a very transparent proof of Takayama's main result is available in (Raa 2017). Both Raa and Katzner show that for budget constrained homothetic preference maximization implies that the change in indirect utility resulting from a change in prices, when the indirect utility function is generated by a utility function which is the logarithm of any linear homogeneous representation of the preferences (and is thus supposed to represent the preferences as well) is equal to the change in consumer's surplus.

Having admitted that the results mentioned in the above paragraph are indeed significant, it is worth noting that:

(i) The work horse of most applicable microeconomics are inverse demand functions which are linear in the own price of the commodity, and it is doubtful if such inverse demand functions can be generated by budget constrained homothetic preference maximization (how long should we keep economic theory locked up in its ivory tower?).
(ii) At a very general level, utility functions are defined on the non-negative orthant of the commodity space and not merely on the strictly positive orthant of it and this would conflict with the logarithmic representation as desired by the authors of the possibility result for most homothetic preferences.
(iii) We are concerned with measuring consumer surplus and not just changes in consumer surplus. There is no guarantee that the consumer's surplus can be defined for demand functions generated by arbitrary budget constrained preference maximization when the preferences are homothetic.
(iv) Apart from the fact that many would be very uncomfortable with invoking the concept of willingness to pay for a bundle of goods without referring to any price or income, there seems to be no compelling reason to interpret a certain numerical representation of preferences as a monetary measure of welfare, simply because calculations go through smoothly and provide apparent consistency between two unrelated concepts. The results are quite accidental, even if they are very likely the product of extra-ordinary minds at work, in much the same way that a clock that does not work always shows the correct time twice a day.

In undergraduate and MBA microeconomics courses, what we are really concerned with when we discuss and apply the concept of surplus on the buyers' side of the market is the surplus of a group of consumers rather than that of an individual consumer. Hence, if what we are concerned with is the aggregate demand function of a group of consumers, then the hypothesis of surplus maximization is reasonably safe and beyond obvious dispute, since from the stand-point of microeconomic theory the alternative approach of representing aggregate demand function as that of a budget constrained preference maximizing representative consumer is an entertaining research project whose implications do not influence the classical demand theory of a budget constrained preference maximizing individual consumer at all. On the other hand, if we assume that consumers as individuals are budget constrained preference maximizers while at the same time measuring an individual consumer's welfare by his consumer surplus, then for the sake of a complete theory of consumer demand, we need to show that budget constrained preference maximization implies surplus maximization.

As is well known in consumer demand theory (see for instance Proposition 3.C.1 in (Mas-Colell, Whinston and Green 1995) preferences are 'continuous' on the non-negative orthant of a finite dimensional Euclidean if and only if they are representable by a continuous utility function. The last result in the appendix proves that *for a 'very large class of utility functions', budget constrained utility maximization implies surplus maximization* and hence everything else that it implies. This is a fairly robust result since it holds for a general class of demand functions including many which are routinely applied in applied microeconomics. Hence, our theory does not contradict the Walrasian budget constrained utility maximizing model. In fact, it could be considered to be a generalization of the latter model. Consumer surplus maximization makes sense even for consumers who are not rational. The undergrad micro model of consumer demand is a generalization of the grad level model.

Since, as we have shown before, surplus maximization is equivalent to the willingness to pay being equal to the area under the inverse demand curve up to the amount consumed, consumer surplus – the traditional measure of consumer welfare – makes sense, even for a consumer who is not 'rational' in the sense of classical consumer demand theory. The scope of consumer surplus maximization is wider than the scope of rational consumer choice theory and the former could be considered to be a generalization of the latter. For the required background on budget constrained preference maximization, nothing beyond chapter 3 of (Katzner 1970) (and not even all of it) is really required. It is worth noting that to obtain our last result we use the concept of marginal rate of substitution as being equivalent to marginal willingness to pay, which is available in chapter 7 of (Hayashi 2015). The reason why our proof sails through is that, unlike (Hayashi 2015), we do not invoke a quasi-linear utility for the purposes of a surplus maximization exercise.

The reason why we feel that our main results are important is because the Hicksian measures of changes in aggregate welfare, such as aggregate EV and aggregate CV, are based on the unobservable private information of consumers, i.e. their preferences over consumption bundles which comprise the non-negative orthant of a finite dimensional Euclidean space. Even assuming that consumers do not face any conflict of interest while stating their preferences, it is one thing to ask a decision maker to rank a finite set of alternatives according to his preferences along a criteria or even provide numerical evaluations of a finite set of alternatives according to some criteria (as in the literature on individual and social/group decision theory in the tradition of Kenneth Arrow and Amartya Sen) and an entirely different matter to ask a decision maker to reveal his preferences accurately over alternatives represented by points in the non-negative orthant of a finite dimensional Euclidean space as in Walrasian consumer demand theory, which we are presently concerned with. The problem in the latter case is that of an individual's inability to articulate his preference rather than the inclination to distort it. It is precisely this problem of acquiring relevant private information from individuals that is not required for the functioning of a free (to the extent that is practical) market economy, one formed the basis of Friedrich Hayek's arguments against central planning in his famous debate with Oscar Lange. Unfortunately Milton Friedman hijacked the debate in his well known book where he referred to a free market economy as capitalism and justified it for what it often fails to deliver, i.e. freedom, rather than its real merit, which is usually a workable and practical resource allocation mechanism. The other method by which preferences can be recovered, at least on the range of the demand function, is by appealing to the Hurwicz–Uzawa Integrability theorem and then solving a system of partial differential equations, to recover a utility function that represents consumer preferences. This, as correctly observed and rigorously established in (Border 2003) is easier said than done. In addition, this would require information about individual demand functions, whereas what are estimated from empirical observations are market demand functions. Market demand functions not only depend on aggregate income but also on the distribution of income and thus viewing it as the demand function of an aggregate/representative consumer may not be meaningful. Hence, the question of applying the Hurwicz–Uzawa Integrabilty theorem to the market demand, may not even be possible from a conceptual point of view. On the other hand, aggregate consumers' surplus can be calculated directly from aggregate demand functions, regardless of whether individual demand functions are those of budget constrained preference maximizers or not. I find it difficult to imagine how compensations based on Hicksian measures of changes in welfare can be implemented in reality, without resorting to significant computational complexity, provided such a computation is even feasible. Incidentally we can define the equivalents of EV and CV in terms of consumer surplus: (a) How much income should be taken away from the consumers at new prices to restore the same surplus to them as before? (b) How much income should be given to the consumers at the old prices to provide them with the same consumer surplus that they get at the new prices? How the answer to these two questions may relate to the difference in the two consumers' surpluses is not apparent to the author as yet.

It should be clear to economic theorists that the scope of our discussion, which encompasses all demand functions that satisfy the (uncompensated) law of demand and the choke price property, is considerably larger than the set of demand functions generated by budget constrained utility maximization and satisfies these two properties. The law of demand is well accepted, including by those who have a nodding acquaintance with economics. The choke price property may require mild additional assumptions in order to be theoretically validated. However, this is no stranger to applied economists. Most research in industrial organization is content with linear demand functions which automatically satisfy the choke price property. At the same time, we should not be under the impression that the choke price property is automatic. It is tempting to conclude that smooth (in the sense of C^∞ over the set of consumption vectors with strictly positive coordinates), strictly increasing and strictly concave utility functions, will necessarily intersect all axes. The reason is that in cases with a little more regularity, which in particular requires that the intersections are not tangential, the choke price property is satisfied for all commodities at all price vector-income pairs. (It is worth noting that the Cobb-Douglas utility functions, although smooth, do not increase strictly; nor do Leontief utility functions which represent the perfect complementarity of the preferences). We now provide an example of a smooth strictly concave and strictly increasing utility function with indifference curves not intersecting the axes.

Consider an economy with a single non-monetary consumption good, only non-negative quantities of which are consumed by an agent, and these quantities are represented by Q and M respectively.

Let $U:\mathbb{R}^2_+ \to \mathbb{R}$ be the utility function $U(Q,M) = 2 - e^{-Q} - e^{-M}$ for all $Q \geq 0$, $M \geq 0$. It is easily verified using standard elementary calculus that U is a smooth strictly concave and strictly increasing utility function. Further $u(0,0) = 0$ and $u(Q,M) \leq 2$ for all Q, $M \geq 0$. Consider the point (Q^1, M^1) such that $Q^1 = M^1 = -\log\left(\frac{1}{4}\right)$. Clearly, $Q^1 = M^1 > 0$ and $U(Q^1,M^1) = 2 - \frac{1}{4} - \frac{1}{4} = \frac{3}{2}$.

Towards a contradiction there exists $M \geq 0$ such that $U(0,M) = \frac{3}{2}$. Then $e^{-M} = -\frac{1}{2} < 0$, leading to a contradiction, since the exponential of no real number can be negative. Hence the indifference curve through (Q^1, M^1) does not intersect the axis along which the consumption of the non-monetary good is measured.

Similarly, it can be shown that this indifference curve does not intersect the axis along which the consumption of money is measured.

In fact, a necessary and sufficient condition for an indifference curve of this utility function to intersect the axes is the value of the utility function along the indifference curve is less than 1. An example of such an indifference curve is the one passing through $\left(-\log\left(\frac{3}{4}\right), -\log\left(\frac{3}{4}\right)\right)$.

It should be emphasized that this paper is concerned with the **logical foundations** of consumer demand theory as taught in a course in microeconomics at the undergraduate level or MBA programs in business schools. Hence, the targeted audience of this paper (apart from some exceptions) comprises teachers of the subject at the appropriate level. The text above informs the reader about the model/context and results we are concerned with, all of which is a comprehensive **teaching note**, relegated to an appendix of the paper. Thus, the potential instructor may use the above text to motivate himself/herself and at the same time inform his/her students as to what the topic concerns i.e. the rehabilitation of consumer's surplus as an exact measure of welfare from the stand-point of cost benefit analysis. Thereafter, the appendix can be referred to for a formal presentation.

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Appendix

Consider an economy comprising of $L \geq 1$ commodities and money and $H \geq 1$ consumers.

Let $p \in \mathbb{R}^L_{++}$ denote a price vector the i^{th} co-ordinate of which denotes the price of i^{th} good and let $I \in \mathbb{R}^H_{++}$ denote the vector of incomes of the H consumers.

A demand function D associates to each price vector income vector pair (p,I) the vector $D(p,I) \in \mathbb{R}^L_+$, where for $i \in \{1,..., L\}$: $D_i(p,I)$ is the maximum amount of commodity i that the consumers want to buy at (p,I). If H = 1, then the demand function is that of a single consumer.

Given a commodity i, let $p_{-i} \in \mathbb{R}^{L-1}_{++}$ be a price vector denoting prices of all commodities other than commodity 'i'.

We assume that for all commodities i and pairs $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^H_{++}$, there exists a price of commodity i, $\overline{p}_i(p_{-i}, I) > 0$ called the **choke price of commodity** i at (p_{-i}, I), such that for all $p_i \geq \overline{p}_i(p_{-i}, I)$, $D_i(p_i,p_{-i}), I) = 0$ and for all $0 < p_i < \overline{p}_i(p_{-i}, I)$, $D_i((p_i,p_{-i}), I) > 0$.

Here (p_i,p_{-i}) is the price vector such that p_i is the price of commodity i, and all other prices are as given in p_{-i}.

For all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^H_{++}$, let $\overline{Q}_i(p_{-i}, I) \in \mathbb{R}_{++} \cup \{+\infty\}$ such that $\{D_i((p_i,p_{-i}), I) \mid p_i > 0\} = [0, \overline{Q}_i(p_{-i}, I))$.

We also assume that for all commodities i and pairs $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^H_{++}$ there exists a continuous function $q_i(.\mid(p_{-i}, I)):[0, \overline{Q}_i(p_{-i}, I)) \to \mathbb{R}_{++}$ called the **inverse demand function of commodity** i, such that:

(i) for all $Q_i \in [0,\overline{Q}_i(p_{-i}, I))$, $D_i(q_i(Q_i\mid(p_{-i}, I)),p_{-i}), I)) = Q_i$;
(ii) for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^H_{++}$ and $Q_i, Q'_i \in [0, \overline{Q}_i(p_{-i}, I))$: $[Q_i > Q'_i]$ implies $[q_i(Q'_i\mid(p_{-i}, I)) > q_i(Q_i\mid(p_{-i}, I))]$.

Property (ii) requires that the demand or inverse demand function for each commodity satisfies the **(uncompensated) Law of Demand**.

For each $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^H_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$, let $W_i(Q_i\mid(p_{-i}, I))$ denote the **maximum that the consumers as an aggregate are willing to pay** for Q_i units of commodity i at (p_{-i}, I).

The **consumers as an aggregate** are said to be **competitive surplus maximizers** in market i, if for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{H}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$, $W_i(Q_i \,|\, (p_{-i}, I)) - Q_i q_i(Q_i \,|\, (p_{-i}, I)) \geq W_i(\xi \,|\, (p_{-i}, I)) - \xi q_i(Q_i \,|\, (p_{-i}, I))$ for all $\xi \in [0, \overline{Q}_i(p_{-i}, I))$.

An equivalent definition of competitive maximizers is the following.

The **consumers as an aggregate** are said to be **competitive surplus maximizers** in market i, if for all $(p, I) \in \mathbb{R}^{L}_{++} \times \mathbb{R}^{H}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$, $W_i(D_i(p,I) \,|\, (p_{-i}, I)) - p_i Q_i \geq W_i(Q_i \,|\, (p_{-i}, I)) - p_i Q_i$.

Proposition 1: Given the assumptions above the consumers as an aggregate are competitive surplus maximizers in market i, if and only if for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{H}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $W_i(Q_i \,|\, (p_{-i}, I)) = \int_0^{Q_i} q_i\left(\xi | (p_{-i}, I)\right) d\xi$.

Proof: Suppose the consumers as an aggregate are competitive surplus maximizers in market i.

From the definition of the inverse demand function it follows that for $\xi \leq Q_i$, if $\pi = q_i\left(\xi | (p_{-i}, I)\right)$, then the maximum amount of commodity i that consumers as an aggregate are willing to consume at $((\pi, p_{-i}), I)$ is ξ.

By hypothesis for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{H}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$, $W_i(Q_i \,|\, (p_{-i}, I)) - Q_i q_i(Q_i \,|\, (p_{-i}, I)) \geq W_i(\xi \,|\, (p_{-i}, I)) - \xi q_i(Q_i \,|\, (p_{-i}, I))$ for all $\xi \in [0, \overline{Q}_i(p_{-i}, I))$.

Assuming the required differentiability of $W_i(. \,|\, (p_{-i}, I))$, we get $\frac{\partial W_i(Q_i \,|\, (p_{-i}, I))}{\partial \xi} \leq q_i(Q_i \,|\, (p_{-i}, I))$ if $Q_i = 0$ and $\frac{\partial W_i(Q_i \,|\, (p_{-i}, I))}{\partial \xi} = q_i(Q_i \,|\, (p_{-i}, I))$ if $Q_i \in (0, \overline{Q}_i(p_{-i}, I))$.

Thus, $W_i(Q_i \,|\, (p_{-i}, I)) = \int_0^{Q_i} q_i\left(\xi | (p_{-i}, I)\right) d\xi$ as was required to be proved.

Now suppose that for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{H}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $W_i(Q_i \,|\, (p_{-i}, I)) = \int_0^{Q_i} q_i\left(\xi | (p_{-i}, I)\right) d\xi$.

Let $(p, I) \in \mathbb{R}^{L}_{++} \times \mathbb{R}^{H}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$.

If $Q_i < D_i(p,I)$, $W_i(D_i(p,I) \,|\, (p_{-i}, I)) - p_i D_i(p,I) = W_i(Q_i \,|\, (p_{-i}, I)) - p_i Q_i + \int_{Q_i}^{D_i(p,I)} [q_i\left(\xi | (p_{-i}, I)\right) - p_i] d\xi > W_i(Q_i \,|\, (p_{-i}, I)) - p_i Q_i$, since $q_i(\xi \,|\, (p_{-I}, I)) > p_i$ for all $\xi < D_i(p)$.

If $Q_i > D_i(p,I)$, $W_i(Q_i \,|\, (p_{-i}, I)) - p_i Q_i = W_i(D_i(p,I) \,|\, (p_{-i}, I)) - p_i D_i(p,I) + \int_{D_i(p,I)}^{Q_i} [q_i\left(\xi | (p_{-i}, I)\right) - p_i] d\xi < W_i(D_i(p,I) \,|\, (p_{-i}, I)) - p_i D_i(p,I)$, since $q_i(\xi \,|\, (p_{-I}, I)) < p_i$ for all $\xi > D_i(p)$.

Thus, $W_i(D_i(p)|(p_{-i}, I)) - p_iQ_i \geq W_i(Q_i|(p_{-i}, I)) - p_iQ_i$ for all $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$. This proves the proposition. Q.E.D.

A simple change of variable argument reveals that for all $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$, $\int_0^{D_i(p,I)} q_i\left(\xi|(p_{-i},I)\right)d\xi = \int_{p_i}^{\bar{p}_i(p_{-i},I)} D_i((\pi_i, p_{-i}),I)d\pi_i + p_iD_i(p,I)$.

Hence we have the following corollary of Proposition 1:

Corollary of Proposition 1: Given the assumptions above the consumers as an aggregate are competitive surplus maximizers in market i, if and only if for all $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$: $W_i(D_i(p,I)|(p_{-i}, I)) = \int_{p_i}^{\bar{p}_i(p_{-i},I)} D_i((\pi_i, p_{-i}),I)d\pi_i + p_iD_i(p,I)$.

For each $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$ and $Q \in \prod_{i=1}^{L}\left[0, \bar{Q}_i(p_{-i},I)\right)$, let $W(Q|(p, I)) = \sum_{i=1}^{L} W_i\left(Q_i,(p_{-i},I)\right)$ denote the **maximum that the consumers as an aggregate are willing to pay** for the consumption bundle Q at (p, I) and let $W(Q|(p, I)) - \sum_{i=1}^{L} p_iQ_i$ be the **competitive surplus** that the consumers derive from the consumption bundle Q at (p,I).

The **consumers as an aggregate** are said to be **competitive surplus maximizers**, if for all $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$ and $Q \in \prod_{i=1}^{L}\left[0, \bar{Q}_i(p_{-i},I)\right)$, $W(D(p)|(p, I)) - \sum_{i=1}^{L} p_iD_i(p,I) \geq W(Q|(p, I)) - \sum_{i=1}^{L} p_iQ_i$.

It is easy to see that the consumers as an aggregate are competitive surplus maximizers **if and only if** they are competitive surplus maximizers in all markets.

Thus, and in view of proposition 1, we have the following theorem.

Proposition 2: (i) Given the assumptions above the consumers as an aggregate are competitive surplus maximizers, if and only if they are competitive surplus maximizers in every market which in turn is true if and only if for all $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$, $i \in \{1, \ldots, L\}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $W_i(Q_i|(p_{-i}, I)) = \int_0^{Q_i} q_i\left(\xi|(p_{-i},I)\right)d\xi$.

(ii) Given the assumptions above the consumers as an aggregate are competitive surplus maximizers, if and only if they are competitive surplus maximizers in every market which in turn is true if and only if for all $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$, $i \in \{1, \ldots, L\}$: $W_i(D_i(p,I)|(p_{-i}, I)) = \int_{p_i}^{\bar{p}_i(p_{-i},I)} D_i((\pi_i, p_{-i}),I)d\pi_i + p_iD_i(p,I)$.

Given $h \in \{1,\ldots,H\}$, $i \in \{1,\ldots,L\}$, let $D^h\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$ denote the demand function of consumer h and let $D^h_i\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}_+$ denote the demand function of consumer h for commodity i.

Given $h \in \{1, \ldots, H\}$, $i \in \{1, \ldots, L\}$ and $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$, let $\bar{p}^h_i(p_{-i}, I^h) > 0$ be the **choke price of commodity** i **for consumer** h at (p_{-i}, I^h). Clearly, $\bar{p}_i(p_{-i}, I) = \max_{h\{1,\ldots,H\}} \bar{p}^h_i(p_{-i}, I^h)$.

Hence, for all $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$, $D^h_i((p_i, p_{-i}), I) = 0$ for all $p_i \geq \bar{p}^h_i(p_{-i}, I^h)$. Further, for all commodities I and $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$, $D_i(p, I) = \sum_{h=1}^{H} D^h_i(p, I)$.

As before, for all $i \in \{1,\ldots,L\}$, $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$, let $\overline{Q}^h_i(p_{-i}, I^h) \in \mathbb{R}_{++} \cup \{+\infty\}$ such that $\{D^h_i((p_i, p_{-i}), I) \mid p_i > 0\} = [0, \overline{Q}^h_i(p_{-i}, I))$.

We assume that for all commodities i and pairs $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ there exists a continuous function $q^h_i(.\mid(p_{-i}, I^h)) \colon [0, \overline{Q}^h_{i\,i}(p_{-i}, I^h)) \to \mathbb{R}_{++}$ called the **inverse demand function of commodity** i **for consumer** h, such that:

(i) for all $Q_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$, $D^h_i(q^h_i(Q^h_i \mid (p_{-i}, I^h)), p_{-i}), I^h)) = Q^h_i$;
(ii) for all $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q^{h\prime}_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$: $[Q^h_i > Q^{h\prime}_i]$ implies $[q^h_i(Q^{h\prime}_i \mid (p_{-i}, I)) > q^h_i(Q^h_i \mid (p_{-i}, I))]$.

Property (ii) requires that the demand or inverse demand function of consumer h, for each commodity satisfies the **Law of Demand**.

For each $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q^h_i \in [0, \overline{Q}^h_i(p_{-i}, I))$, let $W^h_i(Q^h_i \mid (p_{-i}, I^h))$ denote the **maximum that consumer** h **is willing to pay** for Q^h_i units of commodity i at (p_{-i}, I^h).

Consumer h is said to be a **competitive surplus maximizer** in market i, if for all $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q^h_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$, $W^h_i(Q^h_i \mid (p_{-i}, I^h)) - Q^h_i q^h_i(Q_i \mid (p_{-i}, I^h)) \geq W_i(\xi \mid (p_{-i}, I^h)) - \xi q^h_i(Q^h_i \mid (p_{-i}, I))$ for all $\xi \in [0, \overline{Q}_i(p_{-i}, I))$.

An equivalent definition of this concept is the following.

Consumer h is said to be a **competitive surplus maximizer** in market i, if for all $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ and $Q^h_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$, $W^h_i(D^h_i(p, I^h) \mid (p_{-i}, I^h)) - p_i D^h_i(p, I^h) \geq W^h_i(Q^h_i \mid (p_{-i}, I^h)) - p_i Q_i$.

By an argument identical to that used for the inverse market demand function we get the following proposition.

Proposition 3: Given the assumptions above the consumer h is a competitive surplus maximizer in market i, if and only if for all $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q^h_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$: $W^h_i(Q^h_i \mid (p_{-i}, I^h)) = \int_0^{Q^h_i} q^h_i\left(\xi \mid \left(p_{-i}, I^h\right)\right) d\xi$.

Now replicating the argument that lead to the corollary of Proposition 1, we get the following.

Corollary of Proposition 3: Given the assumptions above the consumer h is a competitive surplus maximizer in market i, if and only if for all $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, $W^h_i(D^h_i(p,I^h) \mid (p_{-i}, I^h)) = \int_{p_i}^{\bar{p}_i(p_{-i},I)} D^h_i((\pi_i, p_{-i}), I^h)d\pi_i + p_i D^h_i(p,I^h)$.

Since the willingness to pay of the consumers in any market as an aggregate at any allocation is the sum of the individual willingness to pay, it must be the case that for any commodity i, and $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$, $W_i(D_i(p) \mid (p_{-i}, I)) = \sum_{h=1}^{H} W^h_i(D^h_i(p,I^h) \mid (p_{-i}, I^h))$.

Further, for all $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$ and $i \in \{1,\ldots,L\}$, $\overline{Q}_i(p_{-i}, I) = \sum_{h=1}^{H} \bar{Q}^h_i(p_{-i}, I^h)$ so that $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$ if and only if for all $h \in \{1, \ldots, H\}$ there exists $Q^h_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$ such that $Q_i = \sum_{h=1}^{H} Q^h_i$.

Proposition 4: (i) Given the assumptions above the consumers as an aggregate are competitive surplus maximizers in market i, if and only each and every one of them are individually competitive surplus maximizers in market i(i.e. for all $h \in \{1,\ldots,H\}$consumer h is a competitive maximize in market i) which in turn is true if and only if for all $h \in \{1,\ldots,H\}$, $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q^h_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$: $W^h_i(Q^h_i \mid (p_{-i}, I^h)) = \int_0^{Q^h_i} q^h_i(\xi \mid (p_{-i}, I^h))d\xi$.

(ii) Given the assumptions above the consumers as an aggregate are competitive surplus maximizers in market i, if and only each and every one of them are individually competitive surplus maximizers in market i(i.e. for all $h \in \{1,\ldots,H\}$consumer h is a competitive maximizer in market i) which in turn is true if and only if for all $h \in \{1, \ldots, H\}$, $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, $W^h_i(D^h_i(p,I^h) \mid (p_{-i}, I^h)) = \int_{p_i}^{\bar{p}_i(p_{-i},I)} D^h_i((\pi_i, p_{-i}), I^h)d\pi_i + p_i D^h_i(p,I^h)$.

Given consumer h, $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ and $Q^h \in \prod_{i=1}^{L}\left[0, \bar{Q}^h_i(p_{-i}, I^h)\right)$, let $W^h(Q^h \mid (p, I^h))$ denote the **maximum that consumer** h **is willing to pay** for consumption bundle Q^h. Then $W^h(Q^h \mid (p, I^h)) = \sum_{i=1}^{L} W^h_i(Q^h_i \mid (p_{-i}, I^h))$, where $Q^h = (Q^h_1, \ldots, Q^h_L)$.

Consumer h is said to a **competitive surplus maximizer** if for all $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ and $Q^h \in \prod_{i=}^{L}\left[0, \bar{Q}^h_i(p_{-i}, I^h)\right)$, $W^h(D^h(p) \mid (p, I^h)) - \sum_{i=1}^{L} p_i D^h_i(p, I^h) \geq W^h(Q^h \mid (p, I^h)) - \sum_{i=1}^{L} p_i Q^h_i$.

Proposition 5: (i) Given the assumptions above consumer h is a competitive surplus maximizer, if and only if he is a competitive surplus maximizer in every market which in turn is true if and only if for all $i \in \{1,\ldots,L\}$, $(p_{-i}, I^h) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$:$W^h_i(Q_i \,|\, (p_{-i}, I^h)) = \int_0^{Q_i} q^h_i\left(\xi|\left(p_{-i}, I^h\right)\right)d\xi$. $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$, $i \in \{1, \ldots, L\}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $W_i(Q_i \,|\, (p_{-i}, I)) = \int_0^{Q_i} q_i\left(\xi|\left(p_{-i}, I\right)\right)d\xi$.

(ii) Given the assumptions above the consumer h is a competitive surplus maximizer, if and only if he is a competitive surplus maximizer in every market which in turn is true if and only if for all $i \in \{1, \ldots, L\}$, $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, $W^h_i(D^h_i(p,I^h) \,|\, (p_{-i}, I^h))$ $= \int_{p_i}^{\bar{p}_i(p_{-i},I)} D^h_i((\pi_i, p_{-i}), I^h)d\pi_i + p_i D^h_i(p,I^h)$.

Since for each $(p, I) \in \mathbb{R}^L_{++} \times \mathbb{R}^H_{++}$:$Q \in \prod_{i=1}^{L}\left[0, \bar{Q}_i\left(p_{-i}, I\right)\right)$ if and only if for all $h \in \{1, \ldots, H\}$ there exists $Q^h \in \prod_{i=1}^{L}\left[0, \bar{Q}^h_i\left(p_{-i}, I^h\right)\right)$ such that $Q = \sum_{h=1}^{H} Q^h$ we get the following result.

Theorem 1: Given the assumptions above the following statements are equivalent.

(i) The consumers as an aggregate are surplus maximizers.
(ii) The consumers as an aggregate are surplus maximizers in every market.
(iii) Every consumer is a surplus maximizer.
(iv) Every consumer is a surplus maximize in every market.
(v) For all $i \in \{1, \ldots, L\}$ and $h \in \{1, \ldots, H\}$ and $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, and $Q_i \in [0, \overline{Q}^h_i(p_{-i}, I^h))$: $W^h_i(Q_i \,|\, (p_{-i}, I^h)) = \int_0^{Q_i} q^h_i\left(\xi|\left(p_{-i}, I^h\right)\right)d\xi$.
(vi) For all $i \in \{1,\ldots,L\}$ and $h \in \{1,\ldots,H\}$ and $(p, I^h) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, $W^h_i(D^h_i(p,I^h) \,|\, (p_{-i}, I^h))$ $= \int_{p_i}^{\bar{p}_i(p_{-i},I)} D^h_i((\pi_i, p_{-i}), I^h)d\pi_i + p_i D^h_i(p,I^h)$.

Compatibility with budget constrained utility maximization (preferably for advanced level students): Let us now return to the demand functions that we started the appendix off with and assume that they are generated by a single budget constrained utility maximizing consumer. The utility function $U: \mathbb{R}^{L+1}_{+} \rightarrow \mathbb{R}$ is defined on a consumption set of (L+1) dimensional vectors whose last coordinate is money (saved for other purposes or tomorrow).

We assume that the utility function is continuous, strictly increasing and for each $(p,I) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, D(P,I) is the unique solution of:

Maximize U(Q,M)

Subject to $\sum_{j=1}^{L} p_j Q_j + M \leq I$,

$(Q,M) \in \mathbb{R}^{L+1}_{+}$

The price of money is held fixed at one, i.e. money is the numeraire good and the quantity of money demanded at any $(p,I) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ denoted M(p,I) = $I - \sum_{i=1}^{L} p_i D_i(p,I)$.

Suppose that in addition to what we have already assumed for U, we assume that U is twice continuously differentiable and strictly quasi-concave on $\mathbb{R}^{L+1}_{+}$, with $\frac{\partial U(Q,M)}{\partial Q_i} \geq 0$ and $\frac{\partial U(Q,M)}{\partial M} \geq 0$ for all $i \in \{1, \ldots, L\}$ and $(Q,M) \in \mathbb{R}^{L+1}_{+}$. Here, for $(Q,M) \in \mathbb{R}^{L+1}_{+} \setminus \mathbb{R}^{L+1}_{++}$ and any $i \in L$, $\frac{\partial U(Q,M)}{\partial Q_i} = \lim_{n\to\infty} \frac{\partial U\left(Q^{(n)}, M^{(n)}\right)}{\partial Q_i}$ and $\frac{\partial U(Q,M)}{\partial M} = \lim_{n\to\infty} \frac{\partial U\left(Q^{(n)}, M^{(n)}\right)}{\partial M}$, where $<(Q^{(n)}, M^{(n)}) \mid n \in \mathbb{N}>$ is any sequence in $\mathbb{R}^{L+1}_{++}$, with $\lim_{n\to\infty}\left(Q^{(n)}, M^{(n)}\right) = (Q,M)$.

In addition, we make the following **Demand-Utility Compatibility (DUC)** assumption:

For all $(p,I) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ and $i \in L$, $\frac{\partial U\left(D(p,I), M(p,I)\right)}{\partial Q_i} > 0$ and $\frac{\partial U\left(D(p,I), M(p,I)\right)}{\partial M} > 0$, i.e. **all first partial derivatives of U at chosen points are strictly positive**.

Theorem 2: For a budget constrained utility maximizing consumer, for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $W_i(Q_i \mid (p_{-i}, I)) = \int_0^{Q_i} q_i\left(\xi \mid (p_{-i}, I)\right) d\xi$. Hence the consumer is a surplus maximizer.

Proof: From the definition of the inverse demand function of a budget constrained utility maximizing consumer, it follows that for $\xi \leq Q_i$, if $\pi = q_i\left(\xi \mid (p_{-i}, I)\right)$, then the maximum amount of commodity i that the consumer is willing to consume at $((\pi, p_{-i}), I)$ is ξ.

It is well known that given $(p,I) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ and $i \in L$, the marginal rate of substitution between commodity i and money at (D(p,I), M(p,I)) is equal to p_i.

Hence $p_i = \dfrac{\dfrac{\partial U\big(D(p,I),M(p,I)\big)}{\partial Q_i}}{\dfrac{\partial U\big(D(p,I),M(p,I)\big)}{\partial M}}$.

Given commodity i pair $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$, let $q_i(.\,|\,(p_{-I},I)\colon [0, \overline{Q}_i(p_{-i}, I)) \to \mathbb{R}_{++}$ be the inverse demand function at (p_{-i},I)

Thus, $q_i(Q_i\,|\,(p_{-i}, I)) = \dfrac{\dfrac{\partial U\Big(D\big((q_i(Q_i \mid p_{-i},I),p_{-i}\big),I)),M\big((q_i(Q_i \mid p_{-i},I),p_{-i}\big),I))}{\partial Q_i}}{\dfrac{\partial U\Big(D\big((q_i(Q_i \mid p_{-i},I),p_{-i}\big),I)),M\big((q_i(Q_i \mid p_{-i},I),p_{-i}\big),I))}{\partial M}}$

for all $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$.

Thus, for each $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $q_i(Q_i\,|\,(p_{-i}, I))$ is the rate (or 'speed') at which the consumer is willing to pay money for Q_i units of good i given $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$.

Thus for all $(p_{-i}, I) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}_{++}$ and $Q_i \in [0, \overline{Q}_i(p_{-i}, I))$: $W_i(Q_i\,|\,(p_{-i},I)) = \int_0^{Q_i} q_i\big(\xi|(p_{-i},I)\big)d\xi$.

Hence the consumer is a surplus maximizer. Q.E.D.