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Quaternionic Quantum Mechanics: the Particles, Their *q*-Potentials and Mathematical Electron Model

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Abstract

In this work we show the quaternionic quantum descriptions of physical processes from the Planck to macro scale. The results presented here are based on the concepts of the Cauchy continuum and the elementary cell at the Planck scale. The structurally symmetric quaternion relations and the postulate of the quaternion velocity have been important in the present development. The momentum of the expansion and compression $\dot{u}_0(t, x)$ is the consequence of the scalar term $\sigma_0(t, x)$ in the quaternionic deformation potential. The quaternionic $G_0(m)(\sigma_0 + \hat{\phi})$, vectorial $G_0(m)\hat{\phi}$ and scalar $\mathrm{G}_0(m)\sigma_0$ propagators are used to generate the second order PDE systems for the proton, electron and neutron. A mathematical model of an electron is formulated. It is described by the hyperbolic-elliptic partial differential system of quaternion equations with the initial-boundary conditions. The boundary conditions are generated by the quaternion energy flux that is found with the use of the Gauss theorem, the Cauchy–Riemann derivative and other mathematical formulas. The rigorous assessment of the second order PDE systems allows the proposal of two second order PDE systems for the *u* and *d* quarks from the *up* and *down* groups. It was verified that both the proton and the neutron obey experimental findings and are formed by three quarks. The proton and neutron are formed by the *d-u-u* and *d-d-u* complexes, respectively. The *u* and *d* quarks do not comply with the Cauchy equation of motion. The inconsistencies of the quarks' PDE with the quaternion forms of the Cauchy equation of motion account for their short lifetime and the observed Quarks Chains. That is, they explain the Wilczek phenomenological paradox: *Quarks are Born Free, but everywhere they are in Chains*.

Keywords:
ideal elastic solid, electron, quaternionic potential, vectorial potential, proton, quark chains

1. INTRODUCTION

After developing the Hilbert space formulation of quantum mechanics [1], von Neumann looked into higher mathematical analysis, i.e., rings of operators, in an attempt to get rid of some of the *ad hoc* postulates of quantum mechanics. In 1936, together with Birkhoff, he suggested the Quaternionic Quantum Mechanics (QQM), where wave functions or probability amplitudes are quaternion valued [2]. However, despite this early development, systematic work on the quaternionic extension of quantum mechanics has not yet begun. The essential results relevant to the present paper are by Lanczos. His dissertation was on a quaternionic field theory of the classical electrodynamics [3, 4]. In his derivation of the Dirac equation [5], there is a doubling in the number of solutions and the concepts that still remain at the front of the fundamental theory. These articles were unnoticed by contemporaries; Lanczos abandoned quaternions and never returned to Quaternionic Field Theory (QFT).

Almost immediately it was demonstrated that the Cauchy–Riemann type conditions in the quaternion representation are identical in the shape to vacuum equations of electrodynamics [6] and that the Dirac transition amplitudes are quaternion valued [7]. Christianto derived an original wave equation from the correspondence between the Dirac equation and the Maxwell electromagnetic equations via the biquaternionic representation [8]. The Adler schema of the quaternionizing the quantum mechanics inspired the Harari–Shupe's model for the composite quarks and leptons [9, 10] and the substantial progress in the QQM and QFT [11]. Adler presented a major conceptual advance for the purpose of determining whether a quaternionic Hilbert space is suitable for the unification of the standard model forces with gravitation. He provided an introduction to the problem of formulating quantum field theories and concluded that the QQM may fit into the physics of unification and measurement theory issues [12]. Arbab [13] found a quaternionic wave function consisting of real and scalar functions satisfying the quaternionic momentum eigenvalue equation. More recently in [14] he proposed a quaternionic commutator bracket. Giardino considered in [15] the generalization of the imaginary units within the instances of the complex quantum mechanics, and the quaternionic quantum mechanics. The real Hilbert space formalism developed within the QQM was applied by Giardino in [16] to the simple model of the autonomous particle.

The focus here is on quaternion quantum mechanics and the quaternionic field theory. The QQM presented here is ontological in a sense that it starts with being, that are the Cauchy ideal elastic continuum at the macro-scale ($> 10^{-20}$ m) and the "Planck unit cell" at the micro-scale ($\sim 10^{-35}$ m) [17, 18]. The basic categories of being and their relations are governed by the quaternion algebra [18].

The evolution of the Planck–Kleinert Crystal (P-KC) model and the development of the QQM are shown in succeeding papers [18–21]. In this article we present the QQM in its most recent, refined form:

- We use the ontology-based formalism which is based on the Planck–Kleinert crystal concept [17] and the quaternion algebra introduced by Hamilton, Section 1.1.
- The widely used Helmholtz decomposition is employed in the general form in $\mathbb{R}^4$, Section 2.
- All vectors are in the $\mathbb{R}^4$ representation, e.g., the four-velocity is the "new" variable that allows for the symmetrization of the Hamiltonian [22] and the first and second order wave equations.
- Second order PDE systems of the quark particles are proposed.
- The mathematical model of an electron is formulated. It is described by the hyperbolic-elliptic partial differential system of quaternion equations with the initial-boundary conditions. This differential problem is decomposed onto the hyperbolic equation with the Neumann boundary condition on compression and the hyperbolic-elliptic subsystem on a twist with some specific boundary condition concerning rotation. The boundary conditions are generated by the quaternion energy flux that is found with the use of the Gauss theorem, the Cauchy–Riemann derivative and other mathematical formulas.
- Further studies are suggested in order to verify or refute those propositions.

The Cauchy model of the elastic continuum is presented in Section 1.2. We construct a Lagrangian with the use of the Cauchy–Riemann operator and introduce the key new concept, the quaternion valued velocity, in Section 2. Abbreviations used in the text are presented in the Appendix**.**

1.1. Quaternions

The elements of the quaternion algebra used in the QQM and QFT were already presented in [18–20]. The basic definitions and formulas of quaternions and quaternionic functions can be found in [23, 24].

In Hamilton's own words, he created the $\mathbb{R}^4$ analog of complex numbers as the equivalent of the time-space continuum [25]:

Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements; in technical language it may be said to be 'time plus space'... and in this sense it has, or at least involves a reference to, four dimensions.

We demonstrate here that Hamilton's *'time plus space'* is consistent with the Cauchy model of ideal elastic continuum in the quaternionic representation.

The algebra of quaternions $q = q_0 + \hat{q} \in \mathbb{H}$ has all properties of an algebra with the unity. The symbol $\hat{q}$ means a pure imaginary number.

The quaternionic deformation potential, i.e., the deformation four-potential or q-potential is a relativistic function from which the displacement field can be derived. It combines both a compression scalar potential q_0 and a torsion vector potential $\hat{q}$ (twist) into a single quaternion (four-vector) $q = q_0 + \hat{q} \in \mathbb{H}$.

The multiplication of quaternions is noncommutative: $p \cdot q \neq q \cdot p$. The noncommutativity of multiplication is the only property that makes quaternions different from real and complex numbers.

The quaternionic deformation potential is invariant in the sense of Lorentz.

An arbitrary quaternion $q \in \mathbb{H}$ can be written in the algebraic form:

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{1}$$

where i, j, k are called imaginary units and fulfill the relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{2}$$

It is also possible to represent quaternions by some matrices.

The commutator of two elements p and q is defined by the formula:

$$[p, q] = p \cdot q - q \cdot p = 2\hat{p} \times \hat{q} \tag{3}$$

and can be looked at as a measure of noncommutativity. Two quaternions commute, i.e., $[p, q] = 0$ if and only if their vector parts $\hat{p}$ and $\hat{q}$ are collinear.

Quaternion-valued functions describe a lot of useful physical models, e.g., the electric and magnetic fields [26], Section 3. Let $\Omega \subset \mathbb{R}^3$ be a bounded set. The so-called $\mathbb{H}$-valued functions have the form:

$$q(x) = q_0(x) + q_1(x)i + q_2(x)j + q_3(x)k, \quad x = (x_1, x_2, x_3) \in \Omega, \tag{4}$$

where the functions $q_0(x)$, $q_l(x)$, $l = 1, 2, 3$ are real-valued. Properties such as continuity, differentiability, integrability, and so on, have to be possessed by all the components $q_0(x)$, $q_l(x)$, $l = 1, 2, 3$. In this manner, the Banach, Hilbert, and Sobolev spaces of $\mathbb{H}$-valued functions can be defined [26]. In the Hilbert space over $\mathbb{H}$:

$$L^2(\Omega) = \{q : \Omega \to \mathbb{H} \mid \int_\Omega q_0^2 dx < \infty, \ \int_\Omega q_l^2 dx < \infty, \ l = 1, 2, 3\} \tag{5}$$

the inner product can be defined as follows:

$$\langle q_1, q_2 \rangle = \int_\Omega q_1^* \cdot q_2 dx, \quad q_1, q_2 \in L^2(\Omega). \tag{6}$$

The Fourier series, Lebesgue measure, Gelfand triples, Laplace transform, and many others on the vector space of $\mathbb{H}$-valued functions over $\mathbb{H}$ can be defined in a standard way as in the real and complex cases.

In the further part we use the Cauchy–Riemann operator D acting on the quaternion-valued functions $q(x) = q_0(x) + \hat{q}(x)$ of the form:

$$Dq = -\operatorname{div} \hat{q} + \operatorname{grad} q_0 + \operatorname{rot} \hat{q}. \tag{7}$$

Similarly, the functions $q(t, x)$, depending on time t, may be considered.

Moreover, the exponential quaternionic function of quaternionic variables is very useful in applications and has the trigonometrical representation:

$$e^q = e^{q_0}\left(\cos\left|\hat{q}\right| + \frac{\hat{q}}{\left|\hat{q}\right|}\sin\left|\hat{q}\right|\right), \quad q = q_0 + \hat{q} \in \mathbb{H}. \tag{8}$$

Remark 1. *Hurwitz's theorem says that there are only four normed division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the octonions algebra. Lagrange's four-square theorem in number theory states that every non-negative integer is the sum of four integer squares. This theorem may have applications in QQM.*

1.2. The Cauchy dispacement field: the theory ofelasticity and properties at the Planck scale

Cauchy finished the theory of the ideal elastic continuum in 1822 [27], while Poisson [28] immediately after studied the elementary waves. Neumann [29] gave the proof of uniqueness of the solutions of some initial-boundary value problems. The rigorous completeness proof was given by Duhem [30]. The Cauchy theory is the first real, well posed theory of elasticity using the continuum approach, where the macroscopic phenomena are described in terms of field variables [31]: the compression div $\boldsymbol{u}$, and the twist rot $\boldsymbol{u}$. The stress tensor of the ideal elastic continuum is given by

$$\boldsymbol{T} = \lambda\,\mathrm{tr}(\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D}, \tag{9}$$

where tr($\boldsymbol{D}$) is the trace of the strain tensor, $\mathbf{I}$ is the identity matrix and the two moduli of elasticity, λ and μ, are the material-dependent constants. It was shown by Cauchy and Saint Venant that if the particles composing a regular crystal interact pairwise through central forces, then there is an additional symmetry that implies the Poisson ratio 0.25 and equal Lamé's coefficients: $\lambda = \mu$ [32]. The identity grad div $\boldsymbol{u}$ = div grad $\boldsymbol{u}$ + rot rot $\boldsymbol{u}$ implies that the stress tensor becomes:

$$\begin{aligned}\mathrm{div}\,\boldsymbol{T} &= 2\mu\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \mu\,\mathrm{div}\,\mathrm{grad}\,\boldsymbol{u} = \\ &= 3\mu\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u}.\end{aligned} \tag{10}$$

The Cauchy equation of motion generalizes: (1) the Newton laws of motion (the conservation of the linear and angular momenta) to an ideal elastic solid, and (2) the concept of stress in terms of the gradients in the displacement field $\boldsymbol{u}(t, x) \in \mathbb{R}^3$:

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} = 3\mu\,\mathrm{grad}\,\mathrm{div}\boldsymbol{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\boldsymbol{u} + \boldsymbol{F}, \tag{11}$$

where $\boldsymbol{F}$ is the force induced solely by the displacements caused by the entities already present in the elastic continuum. In the following sections we show that the force must be generalized and follows from the Cauchy–Riemann derivative of the deformation potential f caused by the particle waves, i.e., the force in $\mathbb{R}^4$ is the consequence of the 4-potential f produced by the particles.

From Equation (11) the vectorial representation of the energy density in the deformation field can be computed [31, 33]:

$$e = \frac{1}{2}\dot{\boldsymbol{u}}\circ\dot{\boldsymbol{u}} + \frac{3}{2}c^2(\mathrm{div}\boldsymbol{u})^2 + \frac{1}{2}c^2\mathrm{rot}\boldsymbol{u}\circ\mathrm{rot}\boldsymbol{u}, \tag{12}$$

where $\dot{\boldsymbol{u}} = \partial u/\partial t$. The symbol "$\circ$" means the standard inner product in $\mathbb{R}^3$.

In the following we consider the Cauchy continuum with a Face Centered Cubic (FCC) structure. The Young modulus Y describes tensile elasticity which is axial stiffness of the length of a body to deformation along the axis of the applied tensile force. It is related to Lamé's two moduli of elasticity by

$$Y = \mu\frac{(3\lambda + 2\mu)}{(\lambda + \mu)} \overset{\lambda=\mu}{=} 2.5\mu. \tag{13}$$

If l_P denotes the dimension of the FCC elementary cell that consists of the four interacting Planck particles showing the mass m_P, the Planck density equals: $\rho_P = 4m_P/l_P^3$ = const. The computed Planck density, the Young's modulus and the other properties of the Cauchy continuum at the Planck scale are shown in Table 1. We consider the small deformation limit and the negligible density changes. It allows us to assume the constant transverse wave velocity: $c = \sqrt{\mu/\rho_P} = \mathrm{const}$ [31] and Equation (11) becomes:

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{u}}{\partial t^2} = 3\,\mathrm{grad}\,\mathrm{div}\boldsymbol{u} - \mathrm{rot}\,\mathrm{rot}\boldsymbol{u} + \frac{1}{\mu}\boldsymbol{F}. \tag{14}$$

Table 1
The physical constants of the Cauchy continuum (FCC ideal isotropic crystal)

Label used in this work	Planck constants	Symbol for unit	Value	SI unit	Reference
Lattice parameter	Planck length	l_P	$1.616229(38) \times 10^{-35}$	m	[34]
Poisson ratio	–	ν	0.25	–	[32]
Mass of the Planck particle	Planck mass	m_P	$2.176470(51) \times 10^{-8}$	kg	[34]
Duration of the internal process	Planck time	t_P	$5.39116(13) \times 10^{-44}$	s^{-1}	[34]
Transverse wave velocity	Light velocity in vacuum	$c = \frac{l_P}{t_P}$	2.99792458×10^8	$m \cdot s^{-1}$	[34]
Planck density	–	ρ_P	2.062072×10^{97}	$kg \cdot m^{-3}$	[34]
Young's modulus Intrinsic energy density	–	$Y = 2.5\rho_P c^2$	$4.6332447 \times 10^{114}$	$kg \cdot m^{-1} \cdot s^{-2}$	[34]

The Cauchy model (and to the same degree the majority of physical problems) cannot be reduced to vectorial models (the vector product does not permit the formulation of algebra with unity, for example, the division operation is not defined). By acting on Equation (14) by rot and div operators we separate the transverse and the longitudinal processes:

$$\begin{aligned}&\operatorname{div}\left(\frac{1}{c^2}\frac{\partial^2 \boldsymbol{u}}{\partial t^2}=3\nabla(\nabla\circ\boldsymbol{u})-\nabla\times(\nabla\times\boldsymbol{u})+\frac{1}{\mu}\boldsymbol{F}\right)\\&\operatorname{rot}\left(\frac{1}{c^2}\frac{\partial^2 \boldsymbol{u}}{\partial t^2}=3\nabla(\nabla\circ\boldsymbol{u})-\nabla\times(\nabla\times\boldsymbol{u})+\frac{1}{\mu}\boldsymbol{F}\right)\Rightarrow\\&\underset{\nabla\times(\nabla\times\boldsymbol{u}_\phi)=-\Delta\boldsymbol{u}_\phi}{\Rightarrow}\begin{cases}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\operatorname{div}\boldsymbol{u}_0=3\Delta\operatorname{div}\boldsymbol{u}_0+\frac{1}{\mu}\operatorname{div}\boldsymbol{F}\\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\operatorname{rot}\boldsymbol{u}_\phi=\Delta\operatorname{rot}\boldsymbol{u}_\phi+\frac{1}{\mu}\operatorname{rot}\boldsymbol{F},\end{cases}\end{aligned} \tag{15}$$

where $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_\phi$, $\operatorname{rot}\boldsymbol{u}_0 = 0$, $\operatorname{div}\boldsymbol{u}_\phi = 0$. The Cauchy equation of motion combined with the Helmholtz decomposition theorem results in four second-order scalar differential equations, "quattro cluster", and implies the transverse and longitudinal waves in the Cauchy elastic solid.

Remark 2

1. *The mathematical analysis confirms that the Cauchy model is well posed, i.e., it has a unique solution and its behavior changes continuously with respect to the initial conditions* [30].
2. *The Helmholtz decomposition is never unique* [35].
3. *The Hamilton algebra of quaternions and its relation to the four-dimensional space allow combining the Cauchy theory with the electrodynamics, gravity and quantum mechanics.*

2. THEORY

2.1. The Cauchy deformation field in the quaternion representation

The Cauchy classical theory of elasticity is an elegant starting point to show the physical reality and the significance and beauty of quaternions. The Hamilton algebra $\mathbb{H}$ allows the recoupling of compression and twist that are separated in (15). Upon denoting $\sigma_0 = \operatorname{div}\boldsymbol{u}_0 = (\sigma_0, 0, 0, 0)$ and $\hat\phi = \operatorname{rot}\boldsymbol{u}_\phi = (0, \phi_1, \phi_2, \phi_3)$ we get

$$\begin{cases}\frac{\partial^2\sigma_0}{\partial t^2}=3c^2\Delta\sigma_0\\ \frac{\partial^2\hat\phi}{\partial t^2}=c^2\Delta\hat\phi\end{cases}\Leftrightarrow\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma-2c^2\Delta\sigma_0=0. \tag{16}$$

The decomposition $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_\phi$ in Equation (15) results in four equations in Equation (16) and implies the existence of the deformation field $\sigma = \sigma_0 + \hat\phi$ that represents the twist and compression fields as a superposition of real (scalar compression σ_0) and imaginary (twist vector $\hat\phi$) field parts at each point

$$\sigma = \sigma_0 + \hat\phi \in \mathbb{H} \quad \text{and} \quad \sigma^* = \sigma_0 - \hat\phi \in \mathbb{H}. \tag{17}$$

Adding equations in (16) one gets the quaternion form of the Cauchy motion equation

$$\frac{1}{c^2}\frac{\partial^2\sigma}{\partial t^2}-\Delta\sigma-2\Delta\sigma_0=0. \tag{18}$$

Since $\dot{\boldsymbol{u}} \circ \dot{\boldsymbol{u}} = \hat{u} \circ \hat{u} = -\hat{u} \cdot \hat{u} = \hat{u} \cdot \hat{u}^*$, where $\hat{u} = \dot{u}_1 i + \dot{u}_2 j + \dot{u}_3 k$ and $\dot{\boldsymbol{u}} = (\dot{u}_1, \dot{u}_2, \dot{u}_3)$, the overall energy of the deformation field Relation (12) takes the form:

$$e=\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{1}{2}c^2\,\sigma\cdot\sigma^*+c^2\sigma_0^2, \tag{19}$$

where the velocity within the particle wave is given by the Cauchy–Riemann derivative

$$\hat{u}=-\frac{\hbar}{m}D\sigma, \tag{20}$$

where $D\sigma = \operatorname{grad}\sigma_0 + \operatorname{rot}\hat\phi$ because $\operatorname{div}\hat\phi = \operatorname{div}\operatorname{rot}\boldsymbol{u}_\phi = 0$. The overall energy in an arbitrary volume Ω follows from Equation (19):

$$\begin{aligned}E&=\int_\Omega\rho_E(t,x)dx=\\&=\int_\Omega\rho_P\left(\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{1}{2}c^2\,\sigma\cdot\sigma^*+c^2\sigma_0^2\right)dx=mc^2,\end{aligned} \tag{21}$$

where the external potential, e.g., $V(x)$, is omitted.

The kinetic energy in (19) follows from the vectorial form: $\hat{u} \cdot \hat{u}^*/2$, that can be regarded as $\mathbb{R}^3$ representation that does not describe the volume changes. Contrary, the deformations have quaternion representation in $\mathbb{R}^4$. In (17) the quaternion potential, i.e., the deformation four-potential, is defined by:

$$\begin{aligned}&\quad\sigma\quad\;=\quad\sigma_0\quad+\quad\hat\phi.\\&\begin{bmatrix}q\text{-potential,}\\\text{deformation}\end{bmatrix}=\begin{bmatrix}\operatorname{div}\boldsymbol{u}_0\\\text{compression}\end{bmatrix}+\begin{bmatrix}\operatorname{rot}\boldsymbol{u}_\phi\\\text{twist}\end{bmatrix}\end{aligned} \tag{22}$$

The quaternionic velocity $\dot{u} \in \mathbb{H}$ should represent now all the deformation velocities in $\mathbb{R}^4$. In the derivation of the first order quaternion PDEs and in the electron model (Sections 2.6 and 3) we demonstrate the practical application with an example of the particle wave showing the equivalent mass m:

$$\begin{aligned}&\dot{u}\quad=\quad\dot{u}_0\quad+\quad\hat{u}\\&\qquad\quad-\frac{\hbar}{m}\frac{\sigma_0}{l_P}\quad-\frac{\hbar}{m}(\operatorname{grad}\sigma_0+\operatorname{rot}\hat\phi).\\&\begin{bmatrix}\text{velocity of the}\\q\text{-potential changes}\\\text{deformation velocity}\end{bmatrix}=\begin{bmatrix}\text{compression}\\\text{velocity}\end{bmatrix}+\begin{bmatrix}\text{twist velocity}\end{bmatrix}\end{aligned} \tag{23}$$

2.2. The boundary conditions

In order to obtain the boundary conditions we will find the flux S of the energy e. Let $V \subset \Omega$ be any subregion of Ω

with the smooth boundary ∂V. The rate of change of the total energy within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt}\int_V e d^3\boldsymbol{r} = -\int_{\partial V} S \circ \boldsymbol{n} d(\partial V), \tag{24}$$

where $\boldsymbol{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ is the outside normal unit vector to ∂V. It follows from the Gauss theorem that

$$\int_V \frac{\partial e}{\partial t} d^3\boldsymbol{r} = -\int_V \mathrm{div} S d^3\boldsymbol{r}. \tag{25}$$

Thus

$$\frac{\partial e}{\partial t} = -\mathrm{div} S \tag{26}$$

as V was arbitrary.

On the other hand Formula (12) upon differentiation becomes:

$$\frac{\partial e}{\partial t} = \dot{\boldsymbol{u}} \circ \ddot{\boldsymbol{u}} + 3c^2 \mathrm{div}\boldsymbol{u}\, \mathrm{div}\,\dot{\boldsymbol{u}} + c^2 \mathrm{rot}\boldsymbol{u} \circ \mathrm{rot}\,\dot{\boldsymbol{u}} \tag{27}$$

and using Equation (14) with $\boldsymbol{F} = 0$ we have

$$\begin{aligned}\frac{\partial e}{\partial t} &= \dot{\boldsymbol{u}} \circ (3c^2 \mathrm{grad\,div}\boldsymbol{u} - c^2 \mathrm{rot\,rot}\boldsymbol{u}) + 3c^2 \mathrm{div}\boldsymbol{u}\ \mathrm{div}\dot{\boldsymbol{u}} + \\ &+ c^2 \mathrm{rot}\boldsymbol{u} \circ \mathrm{rot}\dot{\boldsymbol{u}}.\end{aligned} \tag{28}$$

Taking into account the identities div $(a\boldsymbol{A}) = \boldsymbol{A} \circ \mathrm{grad}\, a + a\, \mathrm{div}\,\boldsymbol{A}$ and div $(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \circ \mathrm{rot}\,\boldsymbol{A} - \boldsymbol{A} \circ \mathrm{rot}\,\boldsymbol{B}$, Formula (28) becomes:

$$\frac{\partial e}{\partial t} = -\mathrm{div}\left[c^2(\mathrm{rot}\boldsymbol{u}) \times \dot{\boldsymbol{u}} - 3c^2(\mathrm{div}\boldsymbol{u})\dot{\boldsymbol{u}}\right]. \tag{29}$$

Comparing (26) and (29), the energy flux equals $S = c^2\,(\mathrm{rot}\,\boldsymbol{u}) \times \dot{\boldsymbol{u}} - 3c^2(\mathrm{div}\,\boldsymbol{u})\dot{\boldsymbol{u}}$ or in the quaternionic notation

$$\hat{S} = c^2\hat{\phi} \times \hat{u} - 3c^2\sigma_0\hat{u} \tag{30}$$

or equivalently

$$\hat{S} = c^2(\sigma - \sigma_0) \times \hat{u} - 3c^2\sigma_0\hat{u}. \tag{31}$$

Thus, Relation (29) can be written in the form of the quaternionic continuity equation

$$\frac{\partial e}{\partial t} + \mathrm{div}\hat{S} = 0. \tag{32}$$

Because nothing flows over the boundary $\partial\Omega$, we assume that

$$\hat{S} \circ \hat{n} = 0 \tag{33}$$

on $\partial\Omega$, where $\hat{n} = n_1 i + n_2 j + n_3 k$. Equation (33) will be used in the construction of the boundary conditions of an electron model.

2.3. Maxwell equations in the Cauchy continuum

In this section we will show that the Cauchy equation implies the Maxwell equations. We consider System (15) with the nonzero external force field $\boldsymbol{F}$:

$$\begin{cases}\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\mathrm{div}\boldsymbol{u}_0 - 3\Delta\mathrm{div}\boldsymbol{u}_0 = \dfrac{1}{\mu}\mathrm{div}\boldsymbol{F} \\ \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\mathrm{rot}\boldsymbol{u}_\phi - \Delta\mathrm{rot}\boldsymbol{u}_\phi = \dfrac{1}{\mu}\mathrm{rot}\boldsymbol{F}.\end{cases} \tag{34}$$

The force $\boldsymbol{F}$ is due to the deformations div $\boldsymbol{u}$ and rot $\boldsymbol{u}$ caused by the presence of the flux of the external charged particles, i.e., there exists the external displacement field

$$f = f_0 + \hat{f} \in \mathbb{H}. \tag{35}$$

The force field follows from the Cauchy–Riemann derivative and the condition div $\hat{f} = 0$:

$$\boldsymbol{F} = -\mu Df = -\mu\, \mathrm{grad}\, f_0 - \mu\, \mathrm{rot}\, \hat{f}. \tag{36}$$

Combining (34), (36) and the definitions in (22), System (34) becomes:

$$\begin{cases}\dfrac{1}{c^2}\dfrac{\partial^2\sigma_0}{\partial t^2} - 3\Delta\sigma_0 = -\Delta f_0 \\ \dfrac{1}{c^2}\dfrac{\partial^2\hat{\phi}}{\partial t^2} - \Delta\hat{\phi} = \Delta\hat{f}.\end{cases} \tag{37}$$

By noting that the negative f_0 is a result of net inflow of charged particles and it results in the positive change of σ_0 we get: $\Delta f_0 - \Delta\sigma_0 = 0$, System (37) becomes structurally symmetric:

$$\begin{cases}\dfrac{1}{c^2}\dfrac{\partial^2\sigma_0}{\partial t^2} - \Delta\sigma_0 = \Delta f_0 \\ \dfrac{1}{c^2}\dfrac{\partial^2\hat{\phi}}{\partial t^2} - \Delta\hat{\phi} = \Delta\hat{f}.\end{cases} \tag{38}$$

Upon adding equations in (38) one gets quaternionic representation of the Maxwell displacements

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)(\sigma_0 + \hat{\phi}) = \Delta(f_0 + \hat{f}). \tag{39}$$

Introducing the Maxwell potential definitions: $\varphi = \sigma_0 = \mathrm{div}\,\boldsymbol{u}_0$ and $\boldsymbol{A} = \hat{\phi} = \mathrm{rot}\,\boldsymbol{u}_\phi$ where they denote the irrotational scalar and solenoidal vector potentials, we get the four-potential and flux:

$$A = \varphi + \boldsymbol{A} = \varphi + A_1 i + A_2 j + A_3 k, \tag{40}$$

$$\mu J = J_0 + \hat{J} = -\Delta f_0 - \Delta\hat{f} = -\Delta f. \tag{41}$$

The macroscopic version of the Maxwell equations is Formula (39) with the use of (40) and (41):

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)A = \mu J. \tag{42}$$

The microscopic empty space version is as follows. Consider the empty crystal space (without the charged particles) and the irrotational deformations are negligible: $J = 0$. Consequently Relation (42) reduces to

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \text{rot rot}\,\boldsymbol{A} = 0, \tag{43}$$

by the postulate $\varphi = 0$, $\text{div}\,\boldsymbol{A} = 0$. We introduce the definitions:

$$\boldsymbol{E} := -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{44}$$

$$\boldsymbol{H} := \text{rot}\,\boldsymbol{A}. \tag{45}$$

Upon combining (43)–(45) and by taking the rotation in Definition (44): $\text{rot}\,(c^{-1}(\partial\boldsymbol{A}/\partial t)) = -\,\text{rot}\,\boldsymbol{E}$ the Maxwell system for vacuum follows:

$$\begin{cases} \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \text{rot}\,\boldsymbol{H} = 0 \\ \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} + \text{rot}\,\boldsymbol{E} = 0. \end{cases} \tag{46}$$

2.4. Bosons, fermions, quarks and their *q*-potentials

The quaternionic propagators

The coupling of the transverse and the longitudinal waves takes place in the elementary cell of the Cauchy continuum, i.e., at the Planck scale. The quaternionic oscillator controls the acceleration of all the *q*-potential components during the propagation, e.g., in the particle wave in Ω: $\ddot{\sigma}_0$, $\ddot{\phi}_1$, $\ddot{\phi}_2$, $\ddot{\phi}_3$. The function $G_0 \in \mathbb{R}$ will be called the frequency of the oscillator. In the earlier papers, we disregarded that the twists ϕ_1, ϕ_2 and ϕ_3 form the twist vector $\hat{\phi} = \phi_1 i + \phi_2 j + \phi_3 k$ [35] and are controlled by the oscillator G_0. Thus, the relation between the *q*-potential and its scalar component σ_0 will be corrected and consider the two *q*-potential constituents, σ_0 and $\hat{\phi}$ [35]:

$$\left\langle \frac{\partial^2 \sigma}{\partial t^2} \right\rangle = 2\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 8\pi^2 f_P f, \tag{47}$$

and the frequency of the quaternionic oscillator equals

$$G_0(f) = 8\pi^2 f_P f. \tag{48}$$

The particle wave frequency depends on the particle mass, $f = f(m)$, and follows from the $\mathbb{R}^1$ schema, e.g., see Figure 1 in [20]. The sum of moments of all the Planck masses forming the particle wave in Ω (at the arbitrary time t and solely due to the particle wave) equals the momentum of the particle m itself. On the other hand, we may estimate the average momentum of the arbitrary single Planck mass m_P in the elementary cell during the whole particle cycle: $T = f^{-1}$. The complete cycle implies that every Planck mass m_P returns to its initial conditions: $\boldsymbol{u}_P(t) = \boldsymbol{u}_P(t + T)$ and $\dot{\boldsymbol{u}}_P(t) = \dot{\boldsymbol{u}}_P(t + T)$. The overall distance of the Planck mass during the wave cycle T equals $2\pi l_P$. Thus, the average momentum of the Planck mass $\bar{p}(m_P)$ during the particle wave cycle equals

$$\bar{p}(m_P) = m_P \frac{2\pi l_P}{T} = 2\pi m_P l_P f. \tag{49}$$

The momentum of the particle wave m results from the particle wave propagation velocity:

$$p(m) = mc. \tag{50}$$

Both moments must equal: $p(m) = \bar{p}(m_P)$, and the frequency of the particle wave becomes

$$f = \frac{mc}{2\pi m_P l_P}\frac{c}{c} = \frac{mc^2}{2\pi\hbar}, \quad \hbar = m_P c l_P, \tag{51}$$

where upon using the NIST data [36] for the Planck natural units m_P, l_P, t_p and the light velocity c, the constant $\hbar$ introduced in Relation (51) equals the Planck constant [17]. Combining Relations (48), (51) and the definition $f_P = 1/t_P$, the overall frequency of the quaternionic oscillator when the particle mass is known equals

$$G_0(m) = 4\pi\frac{mc^2}{\hbar t_P}. \tag{52}$$

The oscillator might generate the lower frequencies f of the particle wave and the families of propagators

$$G_n = \frac{1}{n}G_0(m) = \frac{1}{n}4\pi\frac{mc^2}{\hbar t_P}, \quad n \in \mathbb{N}, \tag{53}$$

where n can be interpreted as the measure of the propagator volume, e.g., $l_n = nl_P$.

The quaternionic oscillator $G_0(m)$ controls four propagators:

- the scalar I (spin 0), $G_0(m)\sigma \cdot \sigma^*$,
- the scalar II (spin 1/2), $G_0(m)\sigma_0$,
- the vectorial (spin 1/2), $G_0(m)\hat{\phi}$,
- the quaternionic (spin 1/2), $G_0(m)(\sigma_0 + \hat{\phi})$.

The above propagators generate the particle wave and simultaneously, the particles produce different force fields that are represented by the Poisson equation

$$nc^2\Delta\varphi + G_0(m)f = 0, \tag{54}$$

where φ and f are the quaternion valued functions.

Remark 3. *Substituting $mc^2 = E_0$ in (51), the Planck–Einstein relation follows: $E_0 = hf$, where $h = 2\pi\hbar$.*

The bosons

The family of the scalar second order PDE systems of the spin 0 particles results from the schema in (16) and (53). In (55), we show the core set of the three second order PDE and its equivalent, the set of two second order equations: the particle wave and the force field produced by the particle.

This schema can be written as

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0\\ \left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0 \qquad\qquad \Leftrightarrow\\ \left((n-1)\frac{\partial^2}{\partial t^2}-(n-3)c^2\Delta\right)\sigma_0+2G_0(m)\sigma\cdot\sigma^*=0\end{cases} \tag{55}$$
$$\Leftrightarrow\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\tilde{\sigma}_n+2G_0(m)\sigma\cdot\sigma^*=0\\ \left((n-1)\frac{\partial^2}{\partial t^2}-(n-3)c^2\Delta\right)\sigma_0+2G_0(m)\sigma\cdot\sigma^*=0,\end{cases}$$

where $\tilde{\sigma}_n = n\sigma_0 + \hat{\phi}$ and n denotes natural numbers. If $n = 1$, then System (55) results in [19]:

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0\\ \left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0 \qquad \Leftrightarrow\\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0\end{cases} \tag{56}$$
$$\Leftrightarrow\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+2G_0(m)\sigma\cdot\sigma^*=0\\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0.\end{cases}$$

The above two systems are equivalent and have five equations and five unknowns: σ_0, ϕ_1, ϕ_2, ϕ_3 and m. If mass m is unknown it may be treated as the parameter in the Poisson equation above. Equation (56) corresponds to the Klein–Gordon equation, i.e., the spin 0 boson particle.

The second order PDE Systems (55) and (56) comply with the Cauchy equation of motion, i.e., by adding the Poisson and wave equations, Equation (16) results.

The Poisson equation in (56) describes the potential σ_0 of the deformation field

$$c^2\Delta\sigma_0=-G_0(m)\sigma\cdot\sigma^*=-4\pi\frac{mc^2}{\hbar t_P}\sigma\cdot\sigma^*. \tag{57}$$

It can be expressed as a function of the particle mass density $\rho = (m\sigma\cdot\sigma^*)/l_P^3$:

$$c^2\Delta\sigma_0=-4\pi\rho\frac{l_P^3}{m_P t_P^2}=-4\pi\rho G. \tag{58}$$

Using the data in Table 1, the gravitational constant equals: $G = l_P^3/t_P^2 m_P = 6.674082\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}$. The particle mass center equals its wave energy center. The "space-localized" particle is defined in the sense given by the Bodurov definition [37]:

A singularity-free multi-component function $\sigma = (\sigma_0, \phi_1, \phi_2, \phi_3) \in \mathbb{H}$ *of the space* $x = (x_1, x_2, x_3)$ *and time t variables will be called space-localized if* $\|\sigma(t,x)\|_E \to 0$ *sufficiently fast when* $\|x\|_E \to \infty$ *for each t, so that its Hermitean norm*:

$$\|\sigma\|^2=\langle\sigma,\sigma^*\rangle=\int_\Omega\left(\sigma_0^2+\sum_{l=1}^3\phi_l\cdot\phi_l^*\right)dx=\int_\Omega\sigma\cdot\sigma^*dx<\infty \tag{59}$$

remains finite for all time.

The particles formed by the odd number of quarks

The strong coupling is considered in the further sections: $n = 1$ in Relation (53).

We begin with the vectorial potential, where the term $G_0(m)\,\hat{\phi}$ fixes the density of the rate of twist change and is called vectorial propagator:

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0\\ \left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0\end{cases}\Leftrightarrow\begin{cases}\left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma+2G_0(m)\hat{\phi}=0\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0.\end{cases} \tag{60}$$

Upon the rearrangement, the particle wave (electron) and the vectorial Poisson equations are evident. The adding equations in System (60) shows that it complies with the Cauchy equation of motion (16):

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma+2G_0(m)\hat{\phi}=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0,\end{cases}\Rightarrow \tag{61}$$
$$\Rightarrow\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma-2c^2\Delta\sigma_0=0.$$

Note that the wave propagation velocity in System (60) equals the velocity of longitudinal waves in the Cauchy continuum: $c_L=\sqrt{3}c$ [17]. The vectorial Poisson equation in (60) confirms that it is the second order PDE system for electron. The quaternion propagator, $G_0(m)(\sigma_0 + \hat{\phi})$, the vectorial, $G_0(m)\hat{\phi}$, and scalar, $G_0(m)\sigma_0$, propagators are "merged" and form a strongly coupled system. The rearrangement of System (62) is shown below and displays different forms of the second order PDE systems:

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0\\ \left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0\\ c^2\Delta\sigma_0+G_0(m)\sigma_0=0\end{cases}\Leftrightarrow\begin{cases}\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0\\ \left(\frac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0\\ c^2\Delta(\sigma_0-\hat{\phi})+G_0(m)(\sigma_0+\hat{\phi})=0\end{cases}\Leftrightarrow \tag{62}$$
$$\Leftrightarrow\begin{cases}\left(\frac{\partial^2}{\partial t^2}2c^2\Delta\right)\sigma+G_0(m)(\sigma_0+\hat{\phi})=0\\ c^2\Delta\sigma^*+G_0(m)\sigma=0.\end{cases}$$

The comparison of the scalar, vectorial and quaternionic propagators shows that the q-propagator offers the strongest coupling, Equation (62). The quaternionic Poisson equation in (62) reveals that it is the second order PDE system for proton. The sum of equations in (62) shows that system complies with the Cauchy equation (16):

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-2c^2\Delta\right)\sigma+G_0(m)(\sigma_0+\hat{\phi})=0 \\ c^2\Delta(\sigma_0-\hat{\phi})+G_0(m)(\sigma_0+\hat{\phi})=0\end{cases}\Rightarrow \quad (63)$$

$$\Rightarrow\left(\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma-2c^2\Delta\sigma_0=0.$$

Note that the propagation velocity in System (62) exceeds the transverse wave velocity $c'=\sqrt{2}c$.

The quarks

The comparison of Systems (56), (60) and (62) allows for the postulation of second order PDE for the quarks from the *up* and *down* groups. Specifically, the second order system of the u quark from the *up* group equals:

$$\begin{cases}\left(\frac{1}{3}\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\frac{2}{3}G_0(m)\hat{\phi}=0 \\ -c^2\frac{2}{3}\Delta\hat{\phi}-\frac{2}{3}G_0(m)\hat{\phi}=0\end{cases} \quad (64)$$

and the system of the d quark from the *down* group

$$\begin{cases}\frac{1}{3}\frac{\partial^2\sigma}{\partial t^2}+G_0(m)\sigma_0-\frac{1}{3}G_0(m)\hat{\phi}=0 \\ c^2\Delta\left(\sigma_0+\frac{1}{3}\hat{\phi}\right)-G_0(m)\left(\sigma_0-\frac{1}{3}\hat{\phi}\right)=0.\end{cases} \quad (65)$$

The sum of equations in the above quark Systems (64) and (65) does not comply with the Cauchy equation of motion (16) and may indicate their short lifetime. The terms $2/3(G_0(\mathrm{m})\hat{\phi})$ and $-1/3(G_0(\mathrm{m})\hat{\phi})$ in Systems (64) and (65) respectively, are related to the charge (Table 2).

Table 2
The basic properties of the quarks in baryons

Group	Quarks	Charge	Spin
up	*u, c, t*	2/3	1/2
down	*d, s, b*	−1/3	1/2

There are two groups of hadrons: baryons (containing three quarks or three antiquarks); and mesons (containing the quark and an antiquark). In the following we show that Systems (60)–(65) comply with the experimental findings shown in Table 2.

The proton is formed be the two up and the single down quarks: $d-u-u$. Thus by computing the sum of two Systems (64) and one System (65) we may expect the proton, System (62):

$$\begin{cases}\frac{1}{3}\frac{\partial^2\sigma}{\partial t^2}+G_0(m)\sigma_0-\frac{1}{3}G_0(m)\hat{\phi}=0 \\ c^2\Delta\left(\sigma_0+\frac{1}{3}\hat{f}\right)+G_0(m)\left(\sigma_0-\frac{1}{3}\hat{\phi}\right)=0\end{cases}+$$

$$+2\begin{cases}\left(\frac{1}{3}\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\frac{2}{3}G_0(m)\hat{\phi}=0 \\ -c^2\frac{2}{3}\Delta\hat{\phi}+\frac{2}{3}G_0(m)\hat{\phi}=0\end{cases} \quad (66)$$

and the result is identical with Equations (62):

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-2c^2\Delta\right)\sigma+G_0(m)(\sigma_0+\hat{\phi})=0 \\ c^2\Delta(\sigma_0-\hat{\phi})+G_0(m)(\sigma_0+\hat{\phi})=0.\end{cases} \quad (67)$$

The neutron is formed by the one up and the two down quarks: $d-d-u$:

$$2\times\begin{cases}\frac{1}{3}\frac{\partial^2\sigma}{\partial t^2}+G_0(m)\sigma_0-\frac{1}{3}G_0(m)\hat{\phi}=0 \\ c^2\Delta\left(\sigma_0+\frac{1}{3}\hat{\phi}\right)+G_0(m)\left(\sigma_0-\frac{1}{3}\hat{\phi}\right)=0\end{cases}+$$

$$+\begin{cases}\left(\frac{1}{3}\frac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\frac{2}{3}G_0(m)\hat{\phi}=0 \\ -c^2\frac{2}{3}\Delta\hat{\phi}+\frac{2}{3}G_0(m)\hat{\phi}=0\end{cases} \quad (68)$$

and the result is in agreement with neutron System (56):

$$\begin{cases}\frac{\partial^2\sigma}{\partial t^2}-c^2\Delta\sigma+2G_0(m)\sigma_0=0 \\ c^2\Delta\sigma_0+G_0(m)\sigma_0=0.\end{cases} \quad (69)$$

Systems (60), (67) and (69) represent coupled second order PDE's and show the different coupling strengths. The strongest coupling of the proton is related to its enormously long lifetime, Equation (67).

2.5. The quaternionic Schrödinger equation

The vectorial Poisson equation indicates that it is the second order PDE system for an electron. We will apply the schema in System (60) in the integral form of the energy conservation, in Equation (21). We treat the wave as a particle in an arbitrary volume Ω [19]. The energy per mass unit

$$e=\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^* \quad (70)$$

in the volume occupied by the particle wave defines its overall energy

$$E_O=E_p+E_V=\int_\Omega\rho_p e\,dx, \quad (71)$$

where E_p and E_V are energies of the particle and of its force field respectively, ρ_p is the Planck mass density.

The first step in deriving the Schrödinger equation is the choice of the symmetric formula scheme for the particle energy, E_p. Equation (71) can be written in the equivalent form the following schema in System (60). We separate the E_p and E_V terms in the integral formula

$$E_p+E_V=\rho_P\int_\Omega\left(\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*\right)dx\Leftarrow$$

$$\Leftarrow\begin{cases}E_p=\frac{1}{2}\rho_P\int_\Omega(\hat{u}\cdot\hat{u}^*+3c^2\sigma\cdot\sigma^*)dx\\ E_V=\rho_P\int_\Omega(-c^2\hat{\phi}\cdot\hat{\phi}^*)dx.\end{cases} \tag{72}$$

The mass of the particle $m = E_p/c^2$ follows from the particle wave energy in (72):

$$m=\frac{1}{2}\rho_P\int_\Omega\left(3\sigma\cdot\sigma^*+\frac{\hat{u}\cdot\hat{u}^*}{c^2}\right)dx. \tag{73}$$

The terms $3\sigma\cdot\sigma^*$ and $\hat{u}\cdot\hat{u}^*/c^2$ oscillate and depend on the time and position. The symmetric structure of (73) allows normalizing the deformation and mass velocity with respect to the overall particle mass:

$$\begin{aligned}&\int_\Omega\frac{3\rho_P}{m}\sigma\cdot\sigma^*dx=\int_\Omega\psi\cdot\psi^*dx=1,\ \text{where}\ \psi=\sqrt{\frac{3\rho_P}{m}}\sigma,\\ &\int_\Omega\frac{\rho_P}{mc^2}\hat{u}\cdot\hat{u}^*dx=\int_\Omega\psi\cdot\psi^*dx=1,\ \text{where}\ \psi=\sqrt{\frac{\rho_P}{m}}\frac{\hat{u}}{c}.\end{aligned} \tag{74}$$

The quaternionic particle mass density ψ can be called the quaternionic probability because the relation $\int_\Omega\psi\cdot\psi^*dx=1$ in (74) is satisfied. Obviously, the terms $\psi=\sqrt{(3\rho_P/m)}\sigma$ and $\psi\cdot\psi^*$ vary in time.

We analyze the evolution of the wave as in Relations (73) and (74) in the time-invariant potential field, e.g., the particle wave in the field generated by other particles. The overall particle energy is now a sum of the ground and excess energy Q:

$$E=E_p+Q=\int_\Omega\left(\frac{3}{2}\rho_Pc^2\sigma\cdot\sigma^*+\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+V(x)\psi\cdot\psi^*\right)dx. \tag{75}$$

We consider the low excess energies, and the impact of Q on the overall particle mass in (74) is marginal. Thus, Relation (75) becomes:

$$\begin{aligned}E=E_p+Q&=\int_\Omega\left(\frac{1}{2}mc^2\psi\cdot\psi^*+\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+V(x)\psi\cdot\psi^*\right)dx=\\ &=\frac{1}{2}mc^2+\int_\Omega\left(\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+V(x)\psi\cdot\psi^*\right)dx.\end{aligned} \tag{76}$$

Both the E_p and m are constant, thus it is enough to minimize the relation

$$Q=\int_\Omega\left(\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+V(x)\psi\cdot\psi^*\right)dx. \tag{77}$$

The above relation contains two unknowns: $\hat{u}=\partial u/\partial t$ and ψ. By relating the local lattice velocity $\hat{u}$ to the force, specifically to the normalized Cauchy–Riemann derivative of the deformation $l_PD\sigma$, one gets

$$\hat{u}=\frac{\hat{p}}{m}=-\frac{\hbar}{m}D\sigma. \tag{78}$$

By introducing (78) and the normalization (74), Relation (77) generates the functional:

$$Q\left[\psi\right]=\int_\Omega\left(\frac{\hbar^2}{2m}(D\psi)\cdot(D\psi)^*+V(x)\psi\cdot\psi^*\right)dx. \tag{79}$$

The functional $Q[\psi]$, Equation (79), was minimized with respect to a quaternion function, such that ψ satisfies the normalization introduced in the relation. One may follow the schema used in [19]. In simple terms, we seek a differential equation that has to be satisfied by the ψ function to minimize the energies allowed by (79). Given the functional (79), the conditional extreme is found using the Lagrange coefficients method and the Du Bois–Reymond variational lemma [38]. In such a case, ψ satisfies the time-invariant Schrödinger equation satisfied by the particle wave in the ground state of the energy E [19]:

$$-\frac{\hbar^2}{2m}\Delta\psi+V(x)\psi=\lambda\psi, \tag{80}$$

where a constant factor on the right-hand side can be considered as extra energy of the particle in the presence of the field $V = V(x)$. For $E = \lambda$, Equation (80) is clearly the time-independent Schrödinger equation satisfied by the particle in the ground state of the energy E:

$$-\frac{\hbar^2}{2m}\Delta\psi+V(x)\psi=E\psi. \tag{81}$$

2.6. The first order PDE in the Cauchy continuum

The operator quantum mechanics is based on the complex number algebra and matrix algebra. The canonical quantization starts from the classical mechanics and assumes that the point particle is described by a "probabilistic wave function". Dirac applied complex combinations of the displacements and velocities in the linear problem of secondary quantization [39] and replaced the second order Klein–Gordon equation by an array of first order equations. He recognized the problem of medium for the transmission of waves:

It is necessary to set up an action principle and to get a Hamiltonian formulation of the equations suitable for quantization purposes, and for this the aether velocity is required [39].

In this section we follow the Dirac concept. We derive the formulae based on the aether concept, specifically the Cauchy continuum and the quaternionic oscillator $G_\lambda(m)$ for the first order PDE and the separated Planck time scale. The second order particle wave equation, e.g., in the electron PDE System (61), contains two parts:

$$\begin{array}{ccc}
 & \left(\frac{\partial^2}{\partial t^2}-c_L^2\Delta\right)\sigma & + \\
 & \left[\begin{array}{c}\text{second order wave term } \sigma^{\mu}\sigma_{\mu}\text{: variable}\\ \sigma \text{ and constant wave velocity } c_L=\sqrt{3}c\end{array}\right] & + \\
+ & 2G_0(m)\hat{\phi} & = 0. \\
+ & \left[\begin{array}{c}\text{Propagator with oscillator } G_0(m)\\ \text{that runs at two frequencies}\end{array}\right] & = 0
\end{array} \tag{82}$$

We will comply with above schema for the first order PDE:

$$\begin{array}{ccc}
 & \frac{\partial\dot{u}}{\partial t}-c_L D\hat{\dot{u}} & + \\
 & \left[\begin{array}{c}\text{First order wave term } \sigma_{\mu}\text{ : variable}\\ \dot{u}\text{, wave velocity } c_L=\sqrt{3}c\end{array}\right] & + \\
+ & 2G_\lambda(m)\hat{\dot{u}} & = 0. \\
+ & \left[\begin{array}{c}\text{Propagator with oscillator } G_\lambda(m)\\ \text{that runs at particle wave frequency}\end{array}\right] & = 0
\end{array} \tag{83}$$

The first order wave term

We consider System (60) and the relation between the wave velocity and the Cauchy–Riemann derivative equation (78): $D\sigma = -(m/\hbar)\cdot\hat{\dot{u}}$. The expression for the overall particle energy, Equation (72), implies:

- the deformation velocity as the alternative variable:

$$\dot{u} = \dot{u}_0 + \hat{\dot{u}}, \tag{84}$$

where $\dot{u}_0 = -(\hbar/m)\cdot(\sigma_0/l_p)$, $\hat{\dot{u}} = -(\hbar/m)\cdot D\sigma$,

- the longitudinal wave velocity as the wave propagation velocity:

$$c_L = \sqrt{3}c. \tag{85}$$

The motionless particle is considered, thus its wave is at a steady state. The second order time derivative of the q-potential in (83) we express as follows:

$$\frac{\partial^2\sigma}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\sigma}{\partial t}\right). \tag{86}$$

The term in the bracket on the right-hand side is the rate of the q-potential changes. We want to express this term by the new variable and separate the time scales. The rate of changes of the deformation potential $\partial\sigma/\partial t$ is due to the wave propagation within the particle space. The propagation process must follow the extremum principle, i.e., it is the brachistochrone problem [40]. The good example of "local principle" approximation is by Derbes [41].

We know that the wave path fulfills the extremum principle, i.e., the wave path follows its unique trajectory given by the Cauchy–Riemann derivative, $D\sigma$. The trajectory which has the minimum property globally in the whole volume Ω occupied by the particle must have the same property locally. This path grants the shortest possible travelling time for the waves identified in QQM. Considering $\sigma(\boldsymbol{u}(t,x))$ we get the formula

$$\frac{\partial\sigma}{\partial t} = \sum_{l=1}^{3}\frac{\partial\sigma}{\partial u_l}\frac{\partial u_l}{\partial t}.$$

Consequently from (84)–(86) we postulate the following:

$$\begin{cases}\frac{\partial u_l}{\partial t} = c \quad\Rightarrow\quad \frac{\partial\sigma}{\partial t} = \frac{mc_L}{\hbar}\dot{u}\\ \frac{\partial\sigma}{\partial u_l} = \frac{1}{\sqrt{3}}\frac{m}{\hbar}\dot{u}\end{cases} \tag{87}$$

for l = 1, 2, 3. From Relation (84) we get

$$D\sigma = -\frac{m}{\hbar}\hat{\dot{u}} \quad\Rightarrow\quad \Delta\sigma = -\mathrm{DD}\sigma = \frac{m}{\hbar}D\hat{\dot{u}}. \tag{88}$$

Combining Relations (87) and (88), we get the first order particle wave term consistent with the second order Formula (82):

$$\frac{\partial^2\sigma}{\partial t^2} - c_L^2\Delta\sigma = \frac{mc_L}{\hbar}\frac{\partial\dot{u}}{\partial t} - \frac{mc_L^2}{\hbar}D\hat{\dot{u}} = \frac{mc_L}{\hbar}\left(\frac{\partial\dot{u}}{\partial t} - c_L D\hat{\dot{u}}\right). \tag{89}$$

Thus, the first order particle wave term in (83) equals:

$$\frac{\partial\dot{u}}{\partial t} - c_L D\hat{\dot{u}} = 0. \tag{90}$$

The first order quaternionic oscillator

The frequency of the second order quaternionic oscillator results from two time scales in PK-C: $G_0(f) = 8\pi^2 f_P f$. We consider the macro scale only and first order PDE equation thus, by eliminating the Planck frequency from Relation (48), results in the frequency formula of the first order quaternionic oscillator when the particle mass is known:

$$G_\lambda(m) = 4\pi f = 2\frac{mc_L^2}{\hbar} = 6\frac{m}{m_P t_P}. \tag{91}$$

By introducing Relations (90) and (91) in the schema (83), the first order PDE for electron equals

$$\frac{\partial\dot{u}}{\partial t} - c_L D\hat{\dot{u}} - 12\frac{m}{m_P t_P}\hat{\dot{u}} = 0. \tag{92}$$

The electron spin

The energy Relations (72) are symmetrical and, in the case of the electron:

$$\begin{cases}E_{electron\ particle} = \frac{1}{2}\rho_P\int_\Omega(\hat{\dot{u}}\cdot\hat{\dot{u}}^* + c_L^2\sigma\cdot\sigma^*)dx\\ E_{electron\ potential\ field} = \rho_P\int_\Omega(-c^2\hat{\phi}\cdot\hat{\phi}^*)dx.\end{cases} \tag{93}$$

A particle is stable and its energy must be conserved. Thus, it is justified to assume that the constraint $\operatorname{div}\hat{\phi} = 0$ holds for the completed particle cycle. In the static particle we postulate zero dissipation of the twist energy: $\operatorname{div}\hat{\phi} = 0$. It implies the necessity of the spin, $\hat{S}$, the process that will provide the energy conservation:

$$\operatorname{div}\hat{\phi} = \operatorname{div}(\hat{\phi} + \hat{S}) = 0. \tag{94}$$

The equipartition of energy between the twists, $|\hat{\phi}| = |\hat{S}|$, implies the equipartition of moments: $|\hat{\phi}| = |\hat{S}|$. Thus, the overall momentum per mass unit equals: $2|\hat{S}| = \alpha$.

3. MATHEMATICAL MODEL OF AN ELECTRON

We consider the quaternionic system of equations shown in Equation (61):

$$\begin{cases} \dfrac{\partial^2 \sigma}{\partial t^2} - 3c^2 \Delta\sigma + 2G_0(m)\hat{\phi} = 0 \\ c^2\Delta\hat{\phi} - G_0(m)\hat{\phi} = 0. \end{cases} \tag{95}$$

System (95) is equivalent to the real hyperbolic-elliptic system

$$\begin{cases} \dfrac{\partial^2 \sigma_0}{\partial t^2} - 3c^2 \Delta\sigma_0 = 0 \\ \dfrac{\partial^2 \hat{\phi}}{\partial t^2} - 3c^2 \Delta\hat{\phi} + 2G_0(m)\hat{\phi} = 0 \\ c^2\Delta\hat{\phi} - G_0(m)\hat{\phi} = 0 \end{cases} \tag{96}$$

that can be written in the elegant mathematical form

$$\begin{cases} \dfrac{\partial^2 \sigma_0}{\partial t^2} - 3c^2 \Delta\sigma_0 = 0 \\ \dfrac{\partial^2 \hat{\phi}}{\partial t^2} - c^2 \Delta\hat{\phi} = 0 \\ c^2\Delta\hat{\phi} - G_0(m)\hat{\phi} = 0, \end{cases} \tag{97}$$

Each of the three equivalent Systems (95)–(97) obeys the constraints

$$\begin{cases} \operatorname{div}\hat{\phi} = 0 \\ \phi_1 = 2\phi_2. \end{cases} \tag{98}$$

The energy must be conserved. Thus the flux, Equation (31), through the boundary must be zero and we consider the boundary condition in the form

$$(c^2\hat{\phi} \times \hat{u} - 3c^2\sigma_0\hat{u}) \circ \hat{n} = 0, \tag{99}$$

where

$$\hat{u} = -\frac{\hbar}{m}(\nabla\sigma_0 + \operatorname{rot}\hat{\phi}). \tag{100}$$

Combining (99) and (100) results in

$$(-\hat{\phi} \times \nabla\sigma_0 - \hat{\phi} \times \operatorname{rot}\hat{\phi} + 3\sigma_0\nabla\sigma_0 + 3\sigma_0\operatorname{rot}\hat{\phi}) \circ \hat{n} = 0. \tag{101}$$

An electron does not generate the scalar field and we add the condition

$$\nabla\sigma_0 \circ \hat{n} = 0. \tag{102}$$

Consequently the condition (101) becomes

$$(\hat{\phi} \times \operatorname{rot}\hat{\phi} - 3\sigma_0\operatorname{rot}\hat{\phi}) \circ \hat{n} = 0. \tag{103}$$

The twist $\hat{\phi}$ and compression σ_0 in the above differential problem (97) are only coupled at the boundary, Equation (103). In particular, σ_0 affects $\hat{\phi}$ but not vice versa. So, one may formulate two initial-boundary value problems.

3.1. Initial-boundary value problem for compression σ_0

The hyperbolic equation

$$\frac{\partial^2 \sigma_0}{\partial t^2} - 3c^2 \Delta\sigma_0 = 0 \tag{104}$$

with the Neumann boundary condition

$$\nabla\sigma_0 \circ \hat{n} = 0 \tag{105}$$

and the initial conditions

$$\begin{cases} \sigma_0(0,x) = \sigma_{01}(x) \\ \dfrac{\partial\sigma_0}{\partial t}(0,x) = \sigma_{02}(x), \end{cases} \tag{106}$$

where the functions σ_{01} and σ_{02} are given.

3.2. Initial-boundary value problem for twist $\hat{\phi}$

The hyperbolic-elliptic system

$$\begin{cases} \dfrac{\partial^2 \hat{\phi}}{\partial t^2} - c^2 \Delta\hat{\phi} = 0 \\ c^2\Delta\hat{\phi} - G_0(m)\hat{\phi} = 0 \\ \operatorname{div}\hat{\phi} = 0 \\ \phi_1 = 2\phi_2 \end{cases} \tag{107}$$

with the boundary condition

$$(\hat{\phi} \times \operatorname{rot}\hat{\phi} - 3\sigma_0\operatorname{rot}\hat{\phi}) \circ \hat{n} = 0 \tag{108}$$

and the initial conditions

$$\begin{cases} \hat{\phi}(0,x) = \hat{\phi}_{01}(x) \\ \dfrac{\partial\hat{\phi}}{\partial t}(0,x) = \hat{\phi}_{02}(x), \end{cases} \tag{109}$$

where the functions $\hat{\phi}_{01}$ and $\hat{\phi}_{02}$ are given.

The numerical solution of the model will be presented in the next paper.

4. CONCLUSIONS

The presented results are based on the ontological model of the QQM and QFT, i.e., on the Cauchy continuum and the Planck unit cell concepts. The major progress is due to the use of the quaternion algebra and some structurally symmetric quaternion formulas, specifically the postulate of quaternion velocity. It allows the consideration of the momentum of the expanding Cauchy continuum and is the apparent result of the scalar potential $\sigma_0(t, x)$ of the expansion/compression. This idea allows the formulation of the boundary conditions in the electron model. The key new results are listed below:

The quaternionic $G_0(m)(\sigma_0 + \hat{\phi})$, vectorial $G_0(m)\hat{\phi}$ and scalar: $G_0(m)\sigma_0$ propagators are postulated and used to generate the second order PDE systems for the proton, electron and neutron.

The scrupulous assessment of the second order PDE systems allows the postulation of two second order PDE systems for the *u* and *d* quarks from the *up* and *down* groups.

It is shown that both the proton and the neutron adhere to the experimental findings and are formed by three quarks. Namely, the proton and neutron are formed by *d-u-u* and *d – d – u* complexes, respectively. All the above PDE systems comply with the Cauchy equation of motion (16) and can be considered as stable particles.

The *u* and *d* quark systems do not comply with the Cauchy equation of motion. Also experimental efforts to find the individual quarks were without success. Observed were the bound states of the three quarks – the baryons and a quark and an antiquark – the mesons. Wilczek calls it the phenomenological paradox*: Quarks are Born Free, but Everywhere They are in Chains* [42]. The inconsistency of the quarks PDES with the Cauchy equation of motion elucidates the observed *Quarks Chains.*

The principle of special relativity is explained by this model and may well be applied for all practical purposes. The matter almost completely is built up by electromagnetic forces, so we must expect that both Lorentz contraction and time dilatations will be exactly as predicted by the Lorenz transformations. In the light of this prediction, it is unsurprising that the experiments by Michelson, Morley and many others did not reveal the speed of the Earth through the spatial continuum, which at that time was called the ether.

The results indicate the following targets for an immediate future:

- The particles and quarks in the case of higher coupling coefficients: $n \in \mathbb{Z}$, $n \neq 0$.
- The ratios between the constants for the different force fields.
- The rigorous derivation of the first order PDE basing on the extremum principle.

The multivalued coordinate transformation to determine the properties of space with curvature and torsion produced by the second order PDE systems representing the QFT [43].

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APPENDIX

Symbols

$\hbar$	Planck constant in terms of angular frequency
$\mathbb{H}$	quaternion algebra
QFT	quaternion field theory
QQM	quaternion quantum mechanics
PDE	partial differential equation
$c = \frac{l_P}{t_P}$	transverse wave velocity in elastic continuum
$c_L = \sqrt{3}c$	longitudinal wave velocity in elastic continuum
f	frequency of the particle wave
$f_P = \frac{1}{t_P}$	Planck frequency, inverse of the Planck time
$G_0(m)$	frequency of the quaternionic oscillator
$G_0(m)\sigma \cdot \sigma^*$	scalar propagator I
$G_0(m)\sigma_0$	scalar propagator II
$G_0(m)\hat{\phi}$	vectorial propagator
$G_0(m)(\sigma_0 + \hat{\phi})$	quaternionic *q*-potential propagator
h	Planck constant, $h = 2\pi\hbar$
l_P	Planck length
m	equivalent mass of the wave, i.e., mass of the particle
m_P	Planck mass
N	coupling coefficient in the oscillator
T	deformation tensor
$\boldsymbol{u} = (u_1, u_2, u_3)$	displacement in $\mathbb{R}^3$
λ	length of the particle wave
λ, μ	Lamé coefficients
ρ	mass density of the particle $\rho = \rho_E/c^2$, as the equivalent of the energy density ρ_E in the PK-C
ρ_E	density of the deformation energy
$\rho_P = \frac{4m_P}{l_P^3}$	Planck density, i.e., the mass density of the PK-C
$\sigma = (\sigma_0, \phi_1, \phi_2, \phi_3)$	q-potential in $\mathbb{R}^4$, the quaternion deformation potential
σ'	stress tensors
$\sigma^* \cdot \sigma$	strain energy density
$\psi = \sigma\sqrt{\frac{\rho_P}{m}}$	quaternionic particle density, i.e., the particle wave function
$\psi \cdot \psi^*$	probability, i.e., the normalized particle mass density

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