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Guest editorial

Cesarino Bertini*, Piotr Faliszewski**,
Andrzej Paliński***, Izabella Stach***

This special issue of Decision Making in Manufacturing and Services is devoted to Game Theory and Applications and related topics. The origin of the issue is the 10th Spain-Italy-Netherlands Meeting on Game Theory (SING10), which took place from 7–9 July, 2014. The conference was hosted by the Faculty of Management at AGH University of Science and Technology in Kraków, Poland (the main organizer was Izabella Stach).

The history of the SING meetings started at the beginning of the 1980s, with the first meetings held in Italy. Then, meetings were subsequently added in Spain, the Netherlands, and Poland. Nowadays, SING is one of the most important international meetings on game theory organized each year in a European country.

The SING10 meeting in 2014 attracted more than 190 scientists from 5 continents. More about the SING meetings and, in particular, about SING10 can be funded in Gambarelli (2011) and Bertini *et al.* (2014).

The submitted papers (139 presentations, 135 in parallel sessions and 4 in plenary sessions) covered a variety of topics on game theory and its applications. This special issue collects some surveys on recent results in different fields, presented in the conference.

THE STRUCTURE

The first issue, "On Public Values and Power Indices" by Cesarino Bertini and Izabella Stach, analyzes some values and power indices well-defined in the social context, where the goods are public, from different point of view. In particular, they consider

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the Public Good index, the Public Good value, the Public Help index, the König and Bräuninger (or Zipke), and the Rae index. The authors propose an extension of the Public Help index to cooperative games, introduce a new power index with its extension to a game value, and provide some characterizations of the new index and values.

The second one, "Balancing Bilinearly Interfering Elements" by David Carf? and Gianfranco Gambarelli, starts from the consideration that many decisions in various fields of application have to take into account the joined effects of two elements that can interfere with each other. This happens, for example, in Medicine, Agriculture, Public Economics, Industrial Economics, Zootechnics, and so on. When it is necessary to decide about the dosage of such elements, there is sometimes a primary interest for one effect rather than another; more precisely, it may be of interest that the effects of an element are in a certain proportion with respect to the effects of the other. It may be also necessary to take into account minimum quantities that must be assigned. The authors present the solution in closed form for the case in which the function of the effects is bilinear.

The third paper, "Allocating Pooled Inventory According to Contributions and Entitlements" by Yigal Gerchak, considers inventory pooling. Inventory pooling is known to be beneficial when demands are uncertain. But when the retailers are independent, the question is how to divide the benefits of pooling. In particular, the author considers a decentralized inventory-pooling scheme where the retailers' entitlements to allocation depend on their contributions to the pool in case of shortage. Then, the author derives the Nash equilibrium, and specializes it to symmetric cases.

The fourth contribution, "On the Non-Symmetric Nash and Kalai–Smorodinsky Bargaining Solutions" again by Yigal Gerchak, refers that, in some negotiation application areas, the usual assumption that the negotiators are symmetric has been relaxed. In particular, weights have been introduced to the Nash Bargaining Solution to reflect the different powers of the players. In particular, the author analyzes the properties and optimization of the non-symmetric Nash Bargaining Solution and of a non-symmetric Kalai–Smorodinsky Bargaining Solution. Then, the author provides extensive comparative statics and comments on the implications of the concepts in supply-chain coordination contexts.

The following issue, "Interval methods for computing strong Nash equilibria of continuous games" by Bartłomiej Jacek Kubica and Adam Woźniak, considers the problem of seeking strong Nash equilibria of a continuous game. For some games, these points cannot be found analytically, but only numerically. Interval methods provide us with an approach to rigorously verify the existence of equilibria in certain points. A proper algorithm is presented. Parallelization of the algorithm is also considered, and numerical results are presented. As a particular example, the authors consider the game of "misanthropic individuals," a game that might have several strong Nash equilibria, depending on the number of players. Finally, an algorithm presented is able to localize and verify these equilibria.

The sixth paper by Andrzej Paliński presents a model of bank loan repayment as a signaling game with a set of discrete types of borrowers. The type of borrower is the return on investment project. A possibility of renegotiation of the loan agreement leads to an equilibrium in which the borrower adjusts the repaid amount to the liquidation value of its assets from the bank's point of view. In the equilibrium there are numerous pooling equilibrium points with values rising according to the expected liquidation value of the loan. Furthermore, the author proposes a mechanism forcing the borrower to pay all of his return instead of the common liquidation value of subset of types of the borrower.

Last but not least is, the issue of Joanna Zwierzchowska, "Hyperbolicity of systems describing value functions in differential games which model duopoly problems". Based on the Bressan and Shen approach, the author presents the extension of the class of non-zero sum differential games for which value functions are described by a weakly hyperbolic Hamilton-Jacobi system. The considered value functions are determined by a Pareto optimality condition for instantaneous gain functions, for which we compare two methods of the unique choice Pareto optimal strategies. Then, the procedure of applying this approach for duopoly is presented.

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We want to thank all of the authors for their significant contributions and all of the reviewers for their valuable work. A special thanks have to be given to Tadeusz Sawik and Waldemar Kaczmarczyk for their useful suggestions.

The guest editors

Cesarino Bertini
Piotr Faliszewski
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On Public Values and Power Indices

Cesarino Bertini*, Izabella Stach**

Abstract. In this paper, we analyze some values and power indices from a different point of view that are well-defined in the social context where the goods are public. In particular, we consider the Public Good index (Holler, 1982), the Public Good value (Holler and Li, 1995), the Public Help index (Bertini *et al.*, 2008), the König and Bräuninger index (1998) also called the Zipke index (Nevison *et al.*, 1978), and the Rae index (1969). The aims of this paper are: to propose an extension of the Public Help index to cooperative games; to introduce a new power index with its extension to a game value; and to provide some characterizations of the new index and values.

Keywords: cooperative game theory, simple game, values, public values, power indices, public power indices

Mathematics Subject Classification: 91A06, 91A12, 91B12

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1. INTRODUCTION

A value for n-person cooperative games is a function able to represent a reasonable expectation of the sharing of global winnings amongst the players. A power index is a value for a particular class of games called simple games. The power indices approach is widely used to measure a priori voting power of members of a committee. The concept of value was introduced for the first time by Lloyd Stowell Shapley in (1953). The following year, Shapley and Martin Shubik introduced the "Shapley and Shubik power index" (Shapley and Shubik, 1954). Since 1954, numerous remarkable power indices have been introduced in the literature for simple games. These power indices are based on diverse bargaining models and/or axiomatic assumptions. Some indices have been derived from existing values; i.e., the Shapley and Shubik (1954) as well as the Banzhaf (1965) and the Coleman (1971). Other power indices were formulated exclusively for simple games; i.e., the Public Good index (Holler, 1982), the Deegan

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and Packel index (1978), and the Johnston index (1978). In this paper, we analyze some values and power indices well-defined in the social context where the goods are public; e.g., the Public Good index, the Public Good value, the Public Help index (Bertini *et al.*, 2008), the König and Bräuninger index (1998) also called the Zipke index (Nevison *et al.*, 1978), and the Rae index (1969). We also introduce an extension of the Public Help index as a game value and a new power index with its extension as a game value. Some properties of the new proposed index and values are given.

The paper is organized as follows. Section 2 presents notations and preliminary definitions that refer to cooperative games, simple games, and several properties of power indices. The power indices considered in this paper, as well as a new proposed power index, are described in Section 3. Section 4 is devoted to comparing the considered power indices from the point of view of some desirable properties. Section 5 presents the normalized and absolute Public good value and the propositions of the extension of the Public Help index, as well as a new index to the game value. The paper ends with Section 6 devoted to concluding remarks and further developments. The appendix, at the end of the paper, contains proof of the identity that serves to demonstrate that the new index proposed in this paper satisfies the dominance property.

2. NOTATIONS AND PRELIMINARY DEFINITIONS

Let $N = \{1, 2, \ldots, n\}$ be a finite set of *players*. Any subset $S \subseteq N$ is called a *coalition*, N is called the *grand coalition*, and $\emptyset$ is called an *empty coalition*. By $|S|$, we denote the number of members of S: therefore; e.g., $|N| = n$. A *cooperative game* is a pair (N, v) where v: $2^N \rightarrow R$, the *characteristic function*, is a real-valued function from the set of all possible coalitions of players of N to the real number set such that $v(\emptyset) = 0$. For every coalition S, $v(S)$ is called the *worth* of S. A cooperative game v is *monotonic* if $v(S) \leq v(T)$ if $S \subset T \subseteq N$.

If v takes values only in the set $\{0, 1\}$, then it is called a *simple* monotonic game. By S_N, we denote the set of all simple monotonic games on N.

A player $i \in S$, in a simple game v, is *crucial* or *pivotal*, for the coalition S, if $v(S) = 1$ and $v(S \setminus \{i\}) = 0$.

In a simple game, coalitions S with $v(S) = 1$ are called *winning coalitions* and coalitions with $v(S) = 0$ *losing coalitions*. By W (or $W(v)$), we denote the set of all winning coalitions, and by W_i, we denote the set of all winning coalitions to which player i belongs.

If a player does not belong to any winning coalition, then he is called a *zero player*. A *null* game is a simple game such that $v(S) = 0\ \forall S \subseteq N$. Naturally, in any null game, each player is a zero player.

In a *minimal winning coalition*, all players are crucial. By W^m or $W^m(v)$, we denote the set of all minimal winning coalitions in v, and by W_i^m, we denote the set of all minimal winning coalitions to which player i belongs.

Either the family of *winning* coalitions W or the subfamily of *minimal* winning coalitions W^m determines the game.

If a player is not contained in any minimal winning coalition (i.e. $i \notin S \; \forall S \in W^m$), then he is called a *null player*.

A *weighted game* (also called a *weighted majority game*), $[q; w_1, ..., w_n]$ is a simple game $v \in S_N$ with real *weights* $w_i \geq 0 \; \forall i \in N$ and a non-negative *quota* q, $\frac{\sum_{i\in N} w_i}{2} < q \leq \sum_{i\in N} w_i$, such that $v(S) = 1 \Leftrightarrow w(S) = \sum_{i\in S} w_i \geq q$.

A *value* is a function f that assigns a payoff distribution $f(v) \in R^n$ to every cooperative game v. The real number $f_i(v)$ is the "value" of the player $i \in N$ in the game v according to f.

A *power index* is a function $f : S_N > R^n$ that assigns to any simple game v vector $f(v) = (f_1(v), f_2(v), \ldots, f_n(v))$ (or equivalently $f(W) = (f_1(W), f_2(W), \ldots, f_n(W))$). The non-negative real number $f_i(v)$ (or $f_i(W)$) is interpreted as a "power" of the corresponding player $i \in N$.

There are some properties that are desirable postulates of power indices. Below, we quote only: efficiency, non-negativity, null player, symmetry, dominance, and bloc properties.

If $\sum_{i\notin N} f_i(v) = 1$ for all $v \in S_N$, we said that power index f satisfies the *efficiency postulate*. A power index f satisfies the *non-negativity postulate* (or *positivity postulate*) if $f_i(v) \geq 0$ for each $i \in N$ and any $v \in S_N$. A power index f satisfies the *null player postulate* if $f_i(v) = 0$ for each null player $i \in N$ and all $v \in S_N$. If for all $v \in S_N$ and for each $i \in N$ and each permutation $\pi : N \to N$ $f_i(v) = f_{\pi(i)}(\pi(v))$ where $(\pi(v))(S) = v(\pi^{-1}(S))$, then we said that power index f satisfies the *symmetry postulate* (also called *anonymity postulate*). Let $v : [q; w_1, ..., w_n]$ be an arbitrary weighted game. A power index f satisfies the *dominance* (or *local monotonicity*) postulate if $w_i \geq w_j \Rightarrow f_i(v) \geq f_j(v)$ for any distinct players $i, j \in N$. Note that, in the literature for simple games, there is also a stronger version of dominance property (called *D-dominance* or *strong dominance*) than is presented here; see, for example, (Felsenthal and Machover, 1995; Bertini *et al.*, 2013a).

Consider a weighted game $W : [q; w_1, ..., w_n]$. Let i and j be two distinct players in W and j is not null. If players i and j form a bloc $i\&j$ (i.e., a new entity not belonging to N) and operate as a single player, then a new game arises which we denote by $W[i\&j]$. The new game $W[i\&j]$ is obtain from W by removing two players i and j and introducing a new player representing the bloc $i\&j$. The quota q stays as there was in W. Any player $k \in N \backslash \{i, j\}$ is also a player in $W[i\&j]$ with the same weight, and the weight of the bloc is equal to the sum of the weights of players i and j; i.e., $w_{i\&j} = (w_i + w_j)$. A power index f satisfies the *bloc* property if $f_{i\&j}(W[i\&j]) \geq f_i(W)$.

3. POWER INDICES

In this section, we recall the definitions of the Public Good index, the Public Help index, the König and Bräuninger index, and the Rae index. In Section 3.5, we introduce a new index. The indices considered here are based on winning or minimal winning coalitions and were originally formulated only for simple games.

Henceforth, all the games considered are monotonic and not null.

3.1. THE RAE INDEX

The Rae index, R, was introduced by Rae in (1969). The Rae index of a simple game W for player i is defined as follows:

$$R_i(W) = \frac{|\{S : i \in S \in W\}|}{2^n} + \frac{|\{S : i \notin S \notin W\}|}{2^n}$$

We remark that this index is equivalent to the Brams and Lake index (1978); see also (Nevison, 1979; Mercik, 1997). There is an affine relation between the absolute Banzhaf and Rae indices; see Dubey and Shapley (1979). Thus, the Rae index can be given by the following formula:

$$R_i(W) = \frac{1}{2} + \frac{2|W_i| - |W|}{2^n}$$

3.2. THE KÖNIG AND BRÄUNINGER'S INDEX (OR ZIPKE INDEX)

Nevison, Zicht, Schoepke in (1978) introduced a power index under the name Zipke index. Then, König and Bräuninger in (1998) reinvented it. In the literature, this index is also called the inclusiveness index, and it can been seen as a measure of success (see, for example, Laruelle, Valenciano, 2011). The König and Bräuninger index, KB, of a simple game W for a player i is defined by:

$$KB_i(W) = \frac{|W_i|}{|W|}$$

3.3. THE PUBLIC GOOD INDEX

The Public Good index (PGI) was defined by Holler in (1982). The PGI considers the coalition value to be a public good. The (relative) PGI of a simple game W for player $i \in N$ is given by:

$$h_i(W) = \frac{|W_i^m|}{\sum\limits_{j \in N} |W_j^m|}$$

The PGI index is also called the Holler-Packel index due to the axiomatization of Holler and Packel (1983). Napel in (1999), (2001) showed the independence and non-redundancy of the Holler and Packel axioms.

The absolute Public Good index of simple game W for arbitrary player i is defined as follows:

$$\overline{h}_i(W) = |W_i^m|$$

For the extension of the PGI index to a game value, see Section 5.

3.4. THE PUBLIC HELP INDEX θ

Bertini, Gambarelli and Stach in (2008) introduced the Public Help index (PHI) as a modification of the Public Good index. This index considers that, in assigning a power to a given player i, all of the winning coalitions containing player i (unlike the PGI index, which only takes minimal winning coalitions into account). Indeed, sometimes every winning coalition is relevant to the bargaining. The Public Help index, θ of a non-null simple game W for a player $i \in N$ is given by:

$$\theta_i(W) = \frac{|W_i|}{\sum\limits_{j \in N} |W_j|}$$

In the case of a null game W, this index is $\theta_i(W) = 0$ for any player i. In (Bertini *et al.*, 2008) an axiomatic characterization of the PHI θ index was provided. For its generalization to a game value, see Section 5.

The absolute PHI θ of a simple game W for a player $i \in N$ is the same as the absolute KB index, and is defined for a given simple game v and a player $i \in N$ as the number of all winning coalitions containing player i, as follows:

$$\overline{\theta}_i(W) = \overline{KB}_i(W) = |W_i|$$

Note that, after the adequate normalization of the KB index, we obtain the PHI θ index:

$$\frac{KB_i}{\sum\limits_{j \in N} KB_j} = \frac{|W_i|}{\sum\limits_{j \in N} |W_j|} = \theta_i(W)$$

3.5. THE PUBLIC HELP INDEX ξ (PHI ξ)

In this section, we introduce a new power index PHI ξ . The PHI ξ index, like the KB and PHI θ indices, takes into account all winning coalitions, but it assumes that the probability of forming a winning coalition is inversely proportional to its cardinality and that the players divide the spoils equally in a winning coalition. The Public Help index ξ , for a non-null game W and $i \in N$, is defined as follows:

$$\xi_i(W) = \sum_{S \in W_i} \frac{1}{|S| \sum\limits_{T \in W} \frac{1}{|T|}} \frac{1}{|S|} = \frac{1}{\sum\limits_{T \in W} \frac{1}{|T|}} \sum_{S \in W_i} \frac{1}{|S|^2}$$

In the case of a null game W, this index is $\xi_i(W) = 0$ for any player i. Note that each coalition S is formed with probability $\frac{1}{|S| \sum_{T \in W} \frac{1}{|T|}}$, which is inversely proportional to the cardinality of S. Therefore, the PHI ξ index can be seen as a hybrid between the PHI index and the Deegan-Packel index.

Justification for introducing the Public Help index ξ is similar to the PHI θ. In assigning the power to players, both indices consider all winning coalitions, not only the minimal wining coalitions as in the PGI. For this reason, ξ and θ rather describe power relations in the consumption of public goods, whereas the PGI analyzes the production of public goods. In production, one has to take care that free-riding is excluded; that is why the PGI considers minimal winning coalitions and in consumption of public goods you cannot avoid free-riding. That is why the Public Help indices give values even to null players. Moreover, ξ (thanks to its formula) gives more power to the winning coalitions with a lower number of members than θ . Thus, the players who contribute to success of less-numerous coalitions obtain more power, and, as a, consequence, null players obtain less power (see Example 4.2).

The absolute PHI ξ of a game W for player $i \in N$, is given by:

$$\overline{\xi}_i(W) = \sum_{S \in W_i} \frac{1}{|S|^2}$$

4. COMPARISON OF POWER INDICES

In this section, we compare the *KB*, PGI, PHI θ, PHI ξ, and Rae power indices, taking into account:

- some desirable properties introduced in Section 2,
- the range of power indices, and
- two examples (4.1 and 4.2).

The König and Bräuninger, Rae, and PHI θ indices are more or less related to the Banzhaf index. The Rae and *KB* indices satisfy the non-negativity, symmetry, dominance, and bloc postulates but violate the efficiency and null player properties. While the range of values of R and KB indices is the same, and is as follows: [0.5; 1]. The Public Good index fulfills the efficiency, non-negativity, symmetry, and null postulates but does not satisfy the dominance and bloc properties. All of the above facts written in this paragraph can be found, for example, in (Bertini *et al.*, 2013a).

The index θ satisfies the efficiency, positivity, and symmetry properties but does not satisfy the null player property. The efficiency and symmetry properties are among the axiomatic characterization of the PHI θ; see (Bertini *et al.*, 2008). Felsenthal and Machover in (1995) demonstrated that, if an index satisfies transfer property, then it also satisfies the dominance postulate. The PHI θ does not satisfy transfer property (see Bertini *et al.*, 2013a), but it satisfies the dominance property (see Theorem 4.1).

Theorem 4.1. *PHI* θ *satisfies the dominance property for any weighted game* $v \in S_N$.

Proof. Consider an arbitrary weighted majority game $v : [q; w_1, ..., w_n]$ and two distinct players $i, j \in N$ with weights w_i, w_j such that $w_i \geq w_j$. Note that W_i (and also W_j) includes a non-empty subset, $W_{i \cup j}$, of all winning coalitions that contain players i and j. Namely, $W_{i \cup j} = \{S \in W : i \in S \wedge j \in S\}$ and $W_{i \cup j} \subset W_i$ and $W_{i \cup j} \subset W_j$. If $w_i \geq w_j$ then for any non-empty coalition $S \in W_j \backslash W_{i \cup j}$ (i.e., $i \notin S$), we have $(S \backslash \{j\}) \cup \{i\} \in W_i$: thus, $|W_i| \geq |W_j|$.

From this, we immediately attain that PHI satisfies the dominance postulate:

$$\theta_i(W) = \frac{|W_i|}{\sum\limits_{k\in N} |W_k|} \geq \frac{|W_j|}{\sum\limits_{k\in N} |W_k|} = \theta_j(W)$$

Kurz in (2014) estimated that the individual power of a player $i \in N$ calculated by θ is in the following range: $\frac{1}{2n} \leq \theta_i(v) \leq \frac{2}{n}$ for any simple game v. In this paper, we show that such an interval is narrower (see Theorem 4.2).

Theorem 4.2. *For any simple game $v \in S_N$, we have $\frac{1}{2n-1} \leq \theta_i(v) \leq \frac{2}{n+1}$ for any $i \in N$.*

Proof. Consider a simple game v with N and an arbitrary player $i \in N$. Let us split the set of all winning coalitions into two distinct sets: $W = W_i \cup (W \backslash W_i)$. Thus, $|W| = |W_i| + |W\backslash W_i|$, $\sum_{j\in N} |W_j| = \sum_{S\in W} |S| = \sum_{S\in W_i} |S| + \sum_{S\in (W\backslash W_i)} |S|$, and:

$$\theta_i(W) = \frac{|W_i|}{\sum\limits_{S\in W_i} |S| + \sum\limits_{S\in (W\backslash W_i)} |S|} \tag{1}$$

We remark that, for any simple game, if $S \in (W\backslash W_i)$ then $S \cup \{i\} \in W_i$ which implies $|W_i| \geq |W\backslash W_i|$ and, as a consequence, also $\sum_{S\in W_i} |S| \geq \sum_{S\in (W\backslash W_i)} |S|$.

Firstly, we demonstrate that the minimal power that an arbitrary player i can obtain in a simple game is equal to $\frac{1}{2n-1}$. The PHI index θ for player i has a minimal value if the denominator of (1) attains a maximal value and the numerator of (1) attains a minimal value. The maximal value of denominator (1) is attain for maximal values of both summands $\sum_{S\in W_i} |S|$ and $\sum_{S\in (W\backslash W_i)} |S|$. The summand $\sum_{S\in (W\backslash W_i)} |S|$ attains a maximal value when for any $S \in W_i$ also $S\backslash\{i\} \in (W\backslash W_i)$.

In this case, we have $\sum_{S\in (W\backslash W_i)} |S| = \sum_{S\in W_i} (|S| - 1) = \sum_{S\in W_i} |S| - |W_i|$, and we can rewrite (1) as follows:

$$\theta_i(W) = \frac{|W_i|}{2\sum\limits_{S\in W_i} |S| - |W_i|} \tag{2}$$

Since $v(N) = 1$ for any non-null game $v \in S_N$, we see that the minimal value of $|W_i|$is equal to 1 for any $i \in N$. Suppose that $|W_i| = 1$. Thus, we have that $\sum_{S\in W_i} |S| = |N| = n$. Now, in (2), substituting 1 for $|W_i|$and n for $\sum_{S\in W_i} |S|$, we conclude:

$$\frac{1}{2n-1} \leq \theta_i(W) \text{ for any } i \in N$$

Now, let us demonstrate that the maximal power that θ can assign to a player $i \in N$ is equal to $\frac{2}{n+1}$. The PHI index θ for player i has a maximal value if the denominator of (1) attains the minimal value and the numerator of (1) attains a maximal value.

The denominator of (1) attains its minimum value if $\sum_{S\in(W\setminus W_i)}|S|=0$. Whereas, $|W_i|$(i.e., numerator of (1)) attains its maximum value when all coalitions with player i are winning (it also means that player i is a dictator). Since there are 2^{n-1} coalitions that contain player i, we see that the maximum value of numerator (1) is equal to $|W_i|=2^{n-1}$ and $\sum_{S\in W_i}|S|=\sum_{k=1}^{n}k\begin{pmatrix}n-1\\k-1\end{pmatrix}$. Applying, for example, the binomial identity $(1+x)^n=\sum_{k=0}^{n}\begin{pmatrix}n\\k\end{pmatrix}x^k\ \forall x\in R$, it could be proven that $\sum_{k=1}^{n}k\begin{pmatrix}n-1\\k-1\end{pmatrix}=(n+1)2^{n-2}$ (for a full demonstration, see the Appendix). Now replacing in (1) $|W_i|$ with 2^{n-1}, $\sum_{S\in W_i}|S|$ with $(n+1)2^{n-2}$ and $\sum_{S\in W\setminus W_i}|S|$ with 0, we immediately attain $\theta_i(W)\leq\frac{|W_i|}{\sum_{S\in W_i}|S|}=\frac{2^{n-1}}{(n+1)2^{n-2}}=\frac{2}{(n+1)}$.

Let us consider the new index ξ introduced in Section 3.5. We will prove that the newly proposed index PHI ξ satisfies the following properties: efficiency, non-negativity, symmetry, and dominance (see Theorems 4.3–4.6).

Theorem 4.3. PHI ξ *satisfies the efficiency postulate:*

$$\sum_{i\in N}\xi_i(W)=\begin{cases}1 & \text{if } W \text{ is not the null game}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let W be a game with a set of players N. If W is a null game, each player is a zero player, so $\sum_{i\in N}\xi_i(W)=0$. While for a non-null game W, we attain:

$$\sum_{i\in N}\xi_i(W)=\sum_{i\in N}\left(\frac{1}{\sum_{T\in W}\frac{1}{|T|}}\sum_{S\in W_i}\frac{1}{|S|^2}\right)=\frac{1}{\sum_{T\in W}\frac{1}{|T|}}\sum_{i\in N}\left(\sum_{S\in W_i}\frac{1}{|S|^2}\right)$$

$$=\frac{1}{\sum_{T\in W}\frac{1}{|T|}}\left(\sum_{S\in W_1}\frac{1}{|S|^2}+\sum_{S\in W_2}\frac{1}{|S|^2}+...+\sum_{S\in W_n}\frac{1}{|S|^2}\right)$$

$$=\frac{1}{\sum_{T\in W}\frac{1}{|T|}}\sum_{S\in W}|S|\frac{1}{|S|^2}=\frac{1}{\sum_{T\in W}\frac{1}{|T|}}\sum_{S\in W}\frac{1}{|S|}=1$$

$$F(x)=\frac{\sqrt{x-1}^2}{x^4}$$

Theorem 4.4. *For any simple game W and for any $i\in N$, we have $\xi_i(W)\geq 0$.*

Proof. The PHI ξ of a non-null simple game W and a player $i\in N$ is always greater than zero. It is consequential that, in any non-null game, at least one winning coalition exists (i.e., grand coalition N). Thus, $N\in W\neq\emptyset$ and $N\in W_i\neq\emptyset$, and, as a consequence, we have $\xi_i(W)=\frac{1}{\sum_{T\in W}\frac{1}{|T|}}\sum_{S\in W_i}\frac{1}{|S|^2}>0$. In the case of a null game $\xi_i(W)=0$ $\forall i\in N$ since, in a null game, all players are zero players.

Theorem 4.5. *For any simple game W,* PHI ξ *satisfies the symmetry postulate.*

Proof. Let us fix a simple game W. It is sufficient to prove that $\xi_i(W) = \xi_{\pi(i)}(\pi(W))$ for each $i \in N$ and all permutations $\pi : N \to N$. In case of a null game, it is straightforward to prove that symmetry holds since, in a null game, each player is zero player. In the case of a non-null game W, we have:

$$\xi_i(W) = \frac{1}{\sum_{T\in W} \frac{1}{|T|}} \sum_{S\in W_i} \frac{1}{|S|^2} = \frac{1}{\sum_{T\in W} \frac{1}{|T|}} \sum_{S\in W_{\pi(i)}} \frac{1}{|S|^2} = \xi_{\pi(i)}(W)$$

Theorem 4.6. *For any simple game W,* PHI ξ *satisfies the dominance postulate.*

Proof. Consider an arbitrary weighted majority game $[q; w_1, \ldots, w_n]$ and two distinct players $i, j \in N$ with weights w_i, w_j respectively such that $w_i \geq w_j$. As in the proof of Theorem 4.1, we can show that, if $w_i \geq w_j$, then $|W_i| \geq |W_j|$, and if $W_j \backslash W_i \neq \emptyset$, then for any winning coalition $S \in (W_j \backslash W_i)$, the coalition $(S\backslash\{j\}) \cup \{i\} \in (W_i \backslash W_j)$, and $|S| = |(S\backslash\{j\}) \cup \{i\}|$. Hence, we have not only that $|W_i \backslash W_j| \geq |W_j \backslash W_i|$, but also $\sum_{S\in W_i\backslash W_j} \frac{1}{|S|^2} \geq \sum_{S\in W_j\backslash W_i} \frac{1}{|S|^2}$; as a consequence, we immediately attain that ξ satisfies the dominance postulate:

$$\begin{aligned}\xi_i(W) = \frac{\sum_{S\in W_i} \frac{1}{|S|^2}}{\sum_{T\in W} \frac{1}{|T|}} &= \frac{\sum_{S\in W_i\backslash W_j} \frac{1}{|S|^2} + \sum_{S\in W_i \bigcap W_j} \frac{1}{|S|^2}}{\sum_{T\in W} \frac{1}{|T|}} \\ &\geq \frac{\sum_{S\in W_j\backslash W_i} \frac{1}{|S|^2} + \sum_{S\in W_i \bigcap W_j} \frac{1}{|S|^2}}{\sum_{T\in W} \frac{1}{|T|}} = \frac{\sum_{S\in W_j} \frac{1}{|S|^2}}{\sum_{T\in W} \frac{1}{|T|}} = \xi_j(W)\end{aligned}$$

Example 4.1. Let us consider a game given by the following characteristic function: $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$, $v(\{2, 3\}) = 0$, $v(\{1, 2\}) = 1$, $v(\{1, 3\}) = 1$, $v(\{1, 2, 3\}) = 1$. In Table 1, we present the payoffs assigned by the Rae, König and Bräuninger, PGI, PHI θ and ξ indices to players in the considered game.

Table 1. *Distribution of power in Example 4.1*

Power index	Player 1	Player 2	Player 3
R	7/8	5/8	5/8
KB	1	2/3	2/3
h	1/2	1/4	1/4
θ	3/7	2/7	2/7
ξ	22/48	13/48	13/48

Example 4.2. Let us consider a game $W = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$. In this game, there are only two minimal winning coalitions: $\{1, 2\}$, $\{1, 3\}$. In Table 2, we present the payments assigned by the considered five power indices to the players.

Table 2. *Distribution of power in Example 4.2*

Power index	Player 1	Player 2	Player 3	Player 4
R	7/8	5/8	5/8	1/2
KB	1	2/3	2/3	1/2
h	1/2	1/4	1/4	0
θ	6/17	4/17	4/17	3/17
ξ	129/324	77/324	77/324	41/324

In Example 4.2 (Table 2), we can observe that PHI ξ violates the null property.

As we can observe in Examples 4.1 and 4.2, the power indices taken into account in this paper split the total wins in different ways and assign different power to the players, but give the same rankings to the players. An interesting fact is that some of these power indices induce the same rankings of players, not only in the considered examples, but also in any simple game. Namely, the König and Bräuninger, PHI θ, and Rae indices rank players in the same way. Moreover, they give the same rankings as the Banzhaf power index, since, for a given game W and a player i, all of these indices (KB, θ, R, and Banzhaf indices) are positive affine transformations of $|W_i|$(see Section 3 and (Bertini *et al.*, 2013a)).

The bloc property is one of the most important properties necessary for power indices to be useful for analysis of block-expansion mechanisms in the decision-making bodies (see, for example, (Felsenthal and Machover, 1995; Jasiński, 2013)). The KB and Rae indices satisfy bloc property, whereas the PGI index does not fulfill this property (see (Bertini *et al.*, 2013a)).

In Table 3, we summarize all results discussed in this section.

Table 3. *Power indices R, KB, h, θ, ξ in comparison*

Property	Power index				
	R	KB	h	θ	ξ
Bloc	*yes*	*yes*	*no*	*?*	*?*
Dominance	*yes*	*yes*	*no*	*yes*	*yes*
Efficiency	*no*	*no*	*yes*	*yes*	*yes*
Non-negativity	*yes*	*yes*	*yes*	*yes*	*yes*
Null player	*no*	*no*	*yes*	*no*	*no*
Symmetry	*yes*	*yes*	*yes*	*yes*	*yes*
Range of power index in a non-null game v, $n \geq 2$	[0.5; 1]	[0.5; 1]	[0; 1]	$\left[\frac{1}{2n-1}; \frac{2}{n+1}\right]$	*?*

By “?”, we denote that it is still an open problem.

5. EXTENSIONS OF THE PHI INDICES TO THE GAME VALUES

In this section, we propose the extensions of the PHI indices (θ and ξ) to general cooperative games. But we start with introducing the Public Good value.

Holler and Li in (1995) extended the PGI index to the Public Good value introducing the concept of a real gaining coalition (RGC).

Definition 5.1. *For a cooperative game (N, v), a subset $S \subseteq N$ is called a* real gaining coalition (RGC) *if, for any $T \subset S$, we have $v(S) - v(T) > 0$.*

Let $RGC(v)$ denote a set of all real gaining coalitions in the game v, and by $RGC_i(v)$ (or RGC_i), we denote a set of all real gaining coalitions containing player i. In a cooperative game, the concept of the real gaining coalition corresponds with the concept of the minimal winning coalition in a simple game. Moreover, for any simple game v, we have $W^m(v) = RGC(v)$.

Definition 5.2. *A player $i \in N$ is a* dummy player *if he does not belong to any $S \in RGC(v)$.*

In a general cooperative game, the concept of a dummy player corresponds with the concept of a null player in a simple game. Therefore, similarly like in a simple game, in any minimal winning coalition, all players contribute to its win, in general cooperative game we have that if and only if all players contribute to the worth of a coalition, the coalition is a RGC.

Holler and Li (1995) defined the Public Good value and its normalized version, only taking into account the payoffs from real gaining coalitions (see Definition 5.3 and 5.4).

Definition 5.3. The Public Good value (or the Holler value) *of cooperative game v for a player $i \in N$ is given by: $\overline{HV}_i(v) = \overline{h}_i(v) = \sum_{S \in RGC_i} v(S)$.*

Normalizing the PGV value to the coalition payoff of the grand coalition N, we obtain a normalized version of PGV.

Definition 5.4. The normalized Public Good value (or the normalized Holler value) *of cooperative game v for a player $i \in N$ is given by: $HV_i(v) = h_i(v) = \frac{\overline{h}_i(v)}{\sum_{j \in N} \overline{h}_j(v)} v(N)$.*

We remark that Holler and Li in (1995) axiomatized the Holler value with four axioms: L1 axiom (efficiency), L2 axiom (mergeability), symmetry, and dummy player.

Following the idea of Holler and Li (1995), we propose the extensions of the PHI indices θ and ξ from simple games to general ones in Sections 5.1 and 5.2. In order to state the definitions of the announced values, we need to first introduce the concept of a gaining coalition (GC).

Definition 5.5. *For a cooperative game (N, v), a subset $S \subseteq N$ is called a* gaining coalition (GC) *if $v(S) > 0$.*

Let $GC(v)$ denote a set of all gaining coalitions in the game v, and by $GC_i(v)$ (or GC_i), we denote a set of all gaining coalitions containing player i. We remark that the concept of the gaining coalition in a general cooperative game is equivalent to the concept of the winning coalition in a simple game. Furthermore, for any simple game v, we have $W(v) = GC(v)$.

5.1. PUBLIC HELP VALUE θ

Let us introduce an extension of the PHI θ into a general game, considering only the payoffs from GCs.

Definition 5.6. The absolute Public Help value $\overline{\theta}$ (*or* the $\overline{KB}$ value) *of cooperative game v for a player $i \in N$ is given by* $\overline{\theta}_i(v) = \sum_{S \in GC_i} v(S)$ *if a coalition* $S \subseteq N$ exists such that $v(S) > 0$, *otherwise* $\overline{\theta}_i(v) = 0$.

We also propose the normalized (to the coalition payoff of grand coalition N) version of the PHV value.

Definition 5.7. The normalized Public Help value θ (*or* the KB value) *of cooperative game v for a player $i \in N$ is given by* $\theta_i(v) = \frac{\overline{\theta}_i(v)}{\sum_{j \in N} \overline{\theta}_j(v)} v(N)$ *if a coalition* $S \subseteq N$ exists such that$v(S) > 0$*, otherwise* $\theta_i(v) = 0$.

Now, we prove that absolute and normalized Public Help values $\overline{\theta}$ and θ satisfy several properties. Namely, in Theorem 5.1, we prove that these values assign non-negative payments to players. In Theorem 5.2, we show that the total gain $v(N)$ is distributed by the normalized PHV θ. And finally, Theorem 5.3 states that $\overline{\theta}$ and θ are symmetric, which means that "symmetric" players received the same payment.

Theorem 5.1. *For any cooperative game (v, N) and for any $i \in N$, we have $\overline{\theta}_i(v) \geq 0$ and $\theta_i(v) \geq 0$.*

Proof. Let (v, N) be a cooperative game. If a coalition $S \subseteq N : v(S) > 0$ does not exist, then directly from Definitions 5.6 and 5.7, we have $\overline{\theta}_i(v) = \theta(v) = 0$. Otherwise, if a coalition $S \subseteq N$ exists such that $v(S) > 0$, then $v(N) > 0$, since v is a monotonic game and either $|GC| > 0$ and $|GC_i| > 0$ for any $i \in N$. Thus, for any $i \in N$, we have $\overline{\theta}_i(v) = \sum_{S \in GC_i} v(S) > 0$ and $\theta_i(v) = \frac{\overline{\theta}_i(v)}{\sum_{j \in N} \overline{\theta}_j(v)} v(N) > 0$, which completes the proof.

Theorem 5.2. *For any cooperative game* (v, N), we have $\sum_{i \in N} \theta_i(v) = v(N)$.

Proof. Let (v, N) be a cooperative game. If a coalition $S \subseteq N$ exists such that $v(S) > 0$, then $v(N) > 0$, since v is a monotonic game and either $|GC| > 0$ and $|GC_i| > 0$ for any $i \in N$. Thus, for any $i \in N$, we have $\sum_{i \in N} \theta_i(v) = \sum_{i \in N} \frac{\sum_{S \in GC_i} v(S)}{\sum_{j \in N} \sum_{S \in GC_j} v(S)} v(N) = v(N) \frac{\sum_{i \in N} \sum_{S \in GC_i} v(S)}{\sum_{j \in N} \sum_{S \in GC_j} v(S)} = v(N)$. Otherwise, if a coalition $S \subseteq N : v(S) > 0$ does not exist, then $v(N) = 0$, and from Definition 5.7, we see that $\theta_i(v) = 0$ which completes the proof.

Theorem 5.3. *For any cooperative game* v, $\overline{\theta}_i(v) = \overline{\theta}_{\pi(i)}(\pi(v))$ and $\theta_i(v) = \theta_{\pi(i)}(\pi(v))$ *for each* $i \in N$and all permutations $\pi : N \to N$.

Proof. Let (v, N) be a cooperative game, i be an arbitrary player, and π a permutation on N. In case of a null game, we have that all players received zero (directly from Definitions 5.6 and 5.7). Thus, the theorem holds. In case of a non-null game, we have: $\overline{\theta}_i(v) = \sum_{S \in GC_i} v(S) = \sum_{S \in GC_{\pi(i)}} v(S) = \overline{\theta}_{\pi(i)}(v)$ and $\theta_i(v) = \frac{\overline{\theta}_i(v)}{\sum_{j \in N} \overline{\theta}_j(v)} v(N) = \frac{\overline{\theta}_{\pi(i)}(v)}{\sum_{\pi(j) \in N} \overline{\theta}_{\pi(j)}(v)} v(N) = \theta_{\pi(i)}(v)$.

5.2. PUBLIC HELP VALUE ξ

Let us introduce an extension of the PHI ξ into the general game. The Public Help value ξ (PHV ξ), like PHV θ , only regards payoffs from GCs.

Definition 5.8. The absolute Public Help value $\overline{\xi}$ *of cooperative game* v *for a player* $i \in N$ *is given by* $\overline{\xi}_i(v) = \sum_{S \in GC_i} \frac{v(S)}{|S|^2}$ *if a coalition* $S \subseteq N$ exists such that$v(S) > 0$, *otherwise* $\overline{\xi}_i(v) = 0$.

We also introduce the normalized (to the coalition payoff of grand coalition N) version of the PHV ξ value:

Definition 5.9. The normalized Public Help value ξ *of cooperative game* v *for a player* $i \in N$ *is given by* $\xi_i(v) = \frac{v(N)}{\sum_{S \in GC} \frac{v(S)}{|S|}} \overline{\xi}_i(v)$ *if a coalition* $S \subseteq N$ exists such that $v(S) > 0$, *otherwise* $\xi_i(v) = 0$.

We state and prove that the absolute and normalized PHVs $\overline{\xi}$ and ξ satisfy the properties considered for $\overline{\theta}$ and θ in Section 5.1. This means the extended values $\overline{\xi}$ and ξ preserve not only non-negativity (Theorem 5.4), but also the symmetry (Theorem 5.6).

Theorem 5.4. *For any cooperative game* v *and for any* $i \in N$, *we have* $\overline{\xi}_i(v) \geq 0$ *and* $\xi_i(v) \geq 0$.

Proof. Let (v, N) be a cooperative game. If a coalition $S \subseteq N : v(S) > 0$ does not exist, then we have $\overline{\xi}_i(v) = \xi(v) = 0$ (directly from Definitions 5.8 and 5.9). Otherwise, if a coalition $S \subseteq N$ exists such that $v(S) > 0$, then $v(N) > 0$, since v is a monotonic game and either $|GC| > 0$ and $|GC_i| > 0$ for any $i \in N$. Thus, for any $i \in N$, we have $\overline{\xi}_i(v) = \sum_{S \in GC_i} \frac{v(S)}{|S|^2} > 0$ and $\xi_i(v) = \frac{v(N)}{\sum_{S \in GC} \frac{v(S)}{|S|}} \sum_{S \in GC} \frac{v(S)}{|S|} > 0$, which completes the proof.

Theorem 5.5. *For any cooperative game* (v, N), *we have* $\sum_{i \in N} \xi_i(v) = v(N)$.

Proof. Let (v, N) be a cooperative game. If a coalition $S \subseteq N$ exists such that $v(S) > 0$, then $v(N) > 0$, since v is a monotonic game and either $|GC| > 0$ and $|GC_i| > 0$ for any $i \in N$.
Thus, for any $i \in N$, we have:

$$\sum_{i\in N} \xi_i(v) = \sum_{i\in N} \frac{v(N)\sum_{S\in GC_i} \frac{v(S)}{|S|^2}}{\sum_{S\in GC} \frac{v(S)}{|S|}} = v(N)\frac{\sum_{i\in N}\sum_{S\in GC_i} \frac{v(S)}{|S|^2}}{\sum_{S\in GC} \frac{v(S)}{|S|}}$$
$$= v(N)\frac{\sum_{S\in GC} |S| \frac{v(S)}{|S|^2}}{\sum_{S\in GC} \frac{v(S)}{|S|}} = v(N)\frac{\sum_{S\in GC} \frac{v(S)}{|S|}}{\sum_{S\in GC} \frac{v(S)}{|S|}} = v(N)$$

Otherwise, if a coalition $S \subseteq N : v(S) > 0$ does not exist, then $v(N) = 0$; and from Definition 5.9, we have that $\xi_i(v) = 0$, which completes the proof.

It is not difficult to prove that both values $\overline{\xi}$ and ξ allocate equal payments to symmetric players (see Theorem 5.6).

Theorem 5.6. *For any cooperative game* v, $\overline{\xi}_i(v) = \overline{\xi}_{\pi(i)}(\pi(v))$ and $\xi_i(v) = \xi_{\pi(i)}(\pi(v))$ where $(\pi(v))(S) = v(\pi^{-1}(S))$ *for every* $i \in N$ and all permutations $\pi : N \to N$.

Proof. Let (v, N) be a cooperative game, i be an arbitrary player, and π a permutation on N. In case of a null game, we see that all players received zero (directly from definitions 5.8 and 5.9). Thus, the theorem holds. In case of a non-null game, we have:

$$\overline{\xi}_i(v) = \sum_{S\in GC_i} \frac{v(S)}{|S|^2} = \sum_{S\in GC_{\pi(i)}} \frac{v(S)}{|S|^2} = \overline{\xi}_{\pi(i)}(v)$$

and

$$\xi_i(W) = \frac{v(N)}{\sum_{S\in GC} \frac{1}{|S|}}\overline{\xi_i}(v) = \frac{v(N)}{\sum_{S\in GC} \frac{1}{|S|}}\overline{\xi}_{\pi(i)}(v) = \xi_{\pi(i)}(v)$$

6. CONCLUSION AND FURTHER DEVELOPMENTS

In this paper, we analyzed from a different point of view some values and power indices well-defined in the social context where the goods are public. We consider the Public Good index (Holler, 1982), the Public Good value (Holler and Li, 1995), the Public Help index (Bertini *et al.*, 2008), the König and Bräuninger index (1998) (see also (Nevison *et al.* 1978; Nevison, 1979)), and the Rae index (1969). The aims of this paper were as follows: to propose an extension of the Public Help index to cooperative games; to introduce a new power index with its extension to a game value; and to provide some characterizations of the new index and values.

It is easy to see that the results shown in this paper are not exhaustive. As developments can be many, we simply indicate that those may be of some interest (in our humble opinion). Namely, the new index (PHI ξ) and two new values (PHV θ and PHV ξ) introduced needed axiomatic derivations. Then, the algorithms for automatic computation of new index and new values could be supplied. We suspect that Public

Help indices θ and ξ satisfy the bloc property, but it is still an open problem. The new power index could be compared to all of the other indices, taking into account other properties; for example, those analyzed in (Felsenthal and Machover, 1998; Bertini *et al.*, 2013a, 2013b). Still, regarding ξ, it might be of some interest to analyze its rankings and compare them with the rankings of other indices.

Last but not least, the new values and PHI indices could be extended to games with *a priori* unions, with incompatibilities, with affinities, or with various probabilities of coalition formation (see e.g., Fragnelli *et al.*, 2009).

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APPENDIX

Here we present proof of the following identity:

$$\sum_{k=1}^{n} k \binom{n-1}{k-1} = (n+1)2^{n-2} \quad \forall n \geq 2 \tag{3}$$

The identity (3) can be proven starting with the binomial identity:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \text{ for any real number } x \text{ and } n \geq 1$$

.

Since $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ for $n \geq k > 0$, the above binomial identity can be rewritten as:

$$(1+x)^n = 1 + n\sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1} x^k \text{ for any real number } x \tag{4}$$

Taking the derivative of the both parts of (4) with respect to x, we attain:

$$n(1+x)^{n-1} = n\sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} \tag{5}$$

Substituting 1 for x, we obtain the following identity:

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1} \quad \forall n \geq 2 \tag{6}$$

Now, taking the derivative of the both parts of (5) with respect to x, we attain $(n-1)(1+x)^{n-2} = \sum_{k=1}^{n}(k-1)\binom{n-1}{k-1}x^{k-2}$. Substituting 1 for x, we have:

$$(n-1)2^{n-2} = \sum_{k=1}^{n}(k-1)\binom{n-1}{k-1} \tag{7}$$

From (7), we can calculate $\sum_{k=1}^{n} k\binom{n-1}{k-1}$; and using identity (6), we conclude:

$$\sum_{k=1}^{n} k\binom{n-1}{k-1} = (n-1)2^{n-2} + \sum_{k=1}^{n}\binom{n-1}{k-1} = (n-1)2^{n-2} + 2^{n-1} = (n+1)2^{n-2}$$

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Balancing Bilinearly Interfering Elements

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Abstract. Many decisions in various fields of application have to take into account the joint effects of two elements that can interfere with each other. This happens, for example, in Medicine (synergic or antagonistic drugs), Agriculture (anti-cryptogamics), Public Economics (interfering economic policies), Industrial Economics (where the demand of an asset can be influenced by the supply of another asset), Zootechnics, and so on. When it is necessary to decide about the dosage of such elements, there is sometimes a primary interest for one effect rather than another; more precisely, it may be of interest that the effects of an element are in a certain proportion with respect to the effects of the other. It may also be necessary to take into account minimum quantities that must be assigned.

In Carfì, Gambarelli and Uristani (2013), a mathematical model was proposed to solve the above problem in its exact form. In this paper, we present a solution in closed form for the case in which the function of the effects is bilinear.

Keywords: bargaining problems, game theory, antagonist elements, interfering elements, optimal dosage, synergies

Mathematics Subject Classification: 91A80; 91A35; 91B26; 90B50

JEL Classification: C71, C72, C78

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1. INTRODUCTION

Many decisions in various fields of application have to take into account the joint effects of two elements that can interfere with each other. This happens, for example, in Medicine (synergic or antagonistic drugs), Agriculture (pesticides), Public Economics (interfering economic policies), Industrial Economics (where the demand of an asset can be influenced by the supply of another asset), Zootechnics, and so on. When it is necessary to decide about the dosage of such elements, there is sometimes a primary

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interest for one effect rather than another; more precisely, it may be of interest that the effects of an element are in a certain proportion with respect to the effects of the other. It may also be necessary to take into account the minimum quantities that should be assigned.

In Carfì, Gambarelli and Uristani (2013), a mathematical model was proposed to solve the above problem in its exact form. In this paper, we present a solution in closed form for the case in which the function of the effects is bilinear.

In the next two sections, the problem will be defined in general terms. In Sections 4 and 5, the case of bilinear interference (free and truncated) will be dealt with. In the following section, an algorithm will be presented for the direct calculation of solutions. At the end, we shall provide some examples of application, and we shall indicate some open problems.

1.1. LITERATURE REVIEW

D. Carfì (2010, 2012a) has introduced a new analytical methodology to examine differentiable normal-form games. He and various collaborators have developed the applicative aspects of the new methodology in several directions, such as Management, Finance, Microeconomics, Macroeconomic, Green Economy, Financial Markets, Industrial Organization, Project Financing and so on – see, for instance, Carfì and Fici (2012), Carfì and Lanzafame (2013), Carfì, Magaudda and Schilirò (2010), Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b), Carfì, Patanè and Pellegrino (2011), Carfì and Perrone (2013, 2012a, 2012b, 2011a, 2011b, 2011c), Carfì and Pintaudi (2012), Carfì and Schilirò (2014a, 2014b, 2013, 2012a,2012b, 2012c, 2012d, 2011a, 2011b, 2011c), Carfì, Musolino, Ricciardello and Schilirò (2012), Carfì, Musolino, Schilirò and Strati (2013), Carfì and Trunfio (2011), Okura and Carfì (2014).

The methodology can suggest useful solutions to a specific Game Theory problem. This analytical framework enables us to incorporate solutions designed "to share the pie fairly". The basic original definition we propose and apply for this methodology is introduced also in Carfì and Schilirò (2014a, 2014b, 2013, 2012a, 2012b, 2012c, 2012d, 2011a, 2011b, 2011c) and Carfì (2012a, 2012b, 2010, 2009a, 2009b, 2009c, 2009d, 2009e, 2008). The method we use to study the payoff space of a normal-form game is devisable in Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b), and Carfì and Schilirò (2014a, 2014b, 2013, 2012a, 2012b, 2012c, 2012d, 2011a, 2011b, 2011c). Other important applications, of the complete examination methodology, are introduced in Agreste, Carfì, and Ricciardello (2012), Arthanari, Carfì and Musolino (2015), Baglieri, Carfì, and Dagnino (2012), Carfì and Fici (2012), Carfì, Gambarelli and Uristani (2013), Carfì and Lanzafame (2013), Carfì, Patanè, and Pellegrino (2011), Carfì and Romeo (2015). A complete treatment of a normal-form game is presented and applied by Carfì (2012a, 2012b, 2010, 2009a, 2009b, 2009c, 2009e, 2008), Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b), Carfì and Perrone (2013, 2012a, 2012b, 2011a, 2011b, 2011c), Carfì and Ricciardello (2013a, 2013b, 2012a, 2012b, 2010, 2009) and Carfì and Schilirò (2014a, 2014b, 2013, 2012a,2012b, 2012c, 2012d, 2011a, 2011b, 2011c). Carfì (2008) proposes a general definition and explains the basic properties of Pareto boundary, which constitutes a fundamental element of the complete analysis of a normal-form game.

2. DEFINITIONS

Let $N = \{1, 2\}$ be a set of labels of the considered interfering elements (i.e., drugs, commodities, and so on) and any related effects resulting from their use (e.g., curing diseases, commodity demand, and so on). From here on, if not otherwise specified, the use of the index "i" will imply "for all $i \in N$", with an analogous use of the index "j".

2.1. THE QUANTITIES

We denote the non-negative quantities of the i-th element as follows:

- Q_i is the quantity effectively used;
- ${Q_i}^{\max}$ is the optimal quantity if the i-th element is used alone;
- ${Q_i}^{\min}$ is the minimum necessary quantity if the i-th element is used alone;
- q_i and ${q_i}^{\min}$ are the corresponding ratios with respect to ${Q_i}^{\max}$:
  - $q_i = Q_i/{Q_i}^{\max}$,
  - ${q_i}^{\min} = {Q_i}^{\min}/{Q_i}^{\max}$.

We call Q, $Q^{\max}$, $Q^{\min}$, q, and $q^{\min}$ the corresponding n-vectors.

It is assumed that ${Q_i}^{\min} < {Q_i}^{\max}$ and ${Q_i}^{\min} \leq Q_i \leq {Q_i}^{\max}$. Given such conditions, q_i and ${q_i}^{\min}$ belong to the interval [0,1].

2.2. THE EFFECTS

Let $e_i(q)$ be a non-negative function expressing the level of the i-th effect when percent quantities q are used. The space of the effects is the set of points $x = (x_1,..., x_n) = e(q)$ according to variations of q. This function should satisfy the conditions that follow.

If no elements are used, then all of the effects are null. If a single element is employed in the optimal dose for use alone, then the level of the relative effect is 1, while the level of the effect for the other is null. Finally, if both elements are employed in the optimal doses for use alone, the resulting effects are given by vector $\delta = (\delta_1, \delta_2)$ with real positive components. In formulae:

- if $q_1 = q_2 = 0$, then $e_1 = e_2 = 0$;
- if $q_1 = 0$ and $q_2 = 1$, then $e_1 = 0$ and $e_2 = 1$;
- if $q_1 = 1$ and $q_2 = 0$, then $e_1 = 1$ and $e_2 = 0$;
- if $q_1 = q_2 = 1$, then $e_1 = \delta_1$ and $e_2 = \delta_2$.

See Figure 1 as an example of an effect's function.

Without loss of generality, we may place the elements in order so that:

$$\delta 1 \leq \delta 2. \tag{1}$$

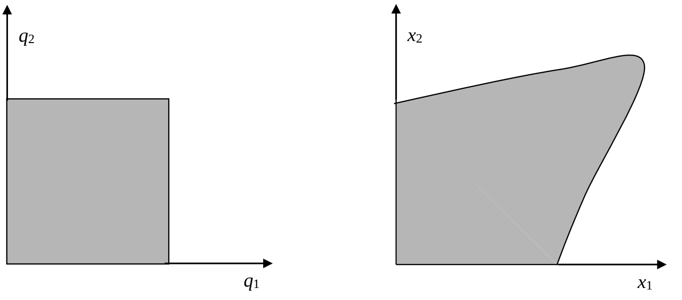

Fig. 1. *Strategy space and payoff space of the game, for $n = 2$*

The effectfunction can be defined directly, according to the faced problem, or can be constructed on the basis of the study cases, using statistical methods and applying suitable adjustments of scale, in order to respect all of the above conditions. In this paper, we study the case in which this function is bilinear: free (Section 4) or truncated (Section 5).

2.3. QUANTITIES AND MINIMUM EFFECTS

We use $e_i^{\min}$ to indicate the minimum necessary level of the i-th effect. This level is derived from the function $e_i(q)$ given $q_i = q_i^{\min}$ and $q_j = 0$ for the other component $j \neq i$. We use $e^{\min}$ to indicate the related 2-dimensional vector.

We assume the minimum necessary level of the i-th effect should not exceed 1 (if $\delta_i \leq 1$) or δ_i(elsewhere). Thus:

$$e_i^{\min} \leq \max\{1, \delta_i\} \tag{2}$$

2.4. THE REQUIRED OPTIMAL RATIOS

We use r to indicate the required optimal ratio between the effects e_1 and e_2. We call R the half-line centered on the origin, the inclination of which is r. For each point x of the feasible set, we use E to indicate the half-line centered on the origin, passing through x.

2.5. THE FEASIBLE PARETO OPTIMAL BOUNDARY

We shall call each point x of the codomain of e which is not jointly improvable a *Pareto optimal effect*, in the sense that if we move from that point in this set to improve the i-th effect, then the other effect necessarily decreases. It is easy to prove that, even here, every Pareto optimal point is a boundary point of the set of effects; we shall, therefore, call the set of Pareto optimal effects the *Pareto optimal boundary*.

The term *feasible Pareto optimal boundary P* is given to the set of the points of the Pareto optimal boundary respecting the conditions $x_i \geq e_i^{\min}$ for all $i \in N$.

3. THE OPTIMIZATION PROBLEM

3.1. THE DATA

The input data of the model is δ, $e^{\min}$, r and the option on the type of bilinear function (free or truncated).

In some applications, we do not directly know the minimal effect $e_i{}^{\min}$ for some element i, while we know the necessary minimal and optimal quantities $Q_i{}^{\min}$ and $Q_i{}^{\max}$. It is thus possible to deduce $q_i{}^{\min}$, which, introduced into the equation $e_i(q)$, gives $e_i{}^{\min}$ (as indicated in Section 2.3).

3.2. THE OBJECTIVE

The problem is to find the set of quantity-vectors q^* such that the corresponding effect vectors $e(q^*)$ belong to the feasible Pareto optimal boundary and are such that the half-lines that join them to the origin form a minimum angle with R.

3.3. EXISTENCE AND UNIQUENESS

If the necessary minimum effects are excessive as a whole, the feasible set is empty; therefore, the problem is without solution. However, for those cases where determining the minimum quantities is open to variations, we have introduced certain indications as to modifications to be used each time. Solution uniqueness is not guaranteed in general, but the various different solutions produce the same effects (payoffs).

3.4. SOLUTION METHODS

Determining the optimal combination of q depends clearly on the form of the effects function $e(q)$. Below, we shall present the solutions for free bilinear functions (Section 4) and for truncated bilinear functions (Section 5) providing closed form formulae and geometrical descriptions. For what concerns cases in which the effect functions are of different types, we refer to Carfì *et al.* (2013).

4. FREE BILINEAR CASE

In such cases, the function $e(q)$ of each effect is defined as follows:

$$e_1 = q_1(1 - q_2) + q_1 q_2 q \delta_1$$

$$e_2 = (1 - q_1)q_2 + q_1 q_2 \delta_2$$

The problem of minimizing the angle between R and E is defined as:

$$\min_{q_1, q_2} \left| \frac{e_2}{e_1} - r \right|$$

We shall examine the various types of interference separately, varying the values of δ under the constraint (1).

We shall represent such types as graphs with corresponding numbers. In each of these graphs, the grey portion indicates the area in which δ can vary, while the bold line indicates the feasible Pareto optimal boundary.

We shall then give the solutions along with the relative steps for achieving them in the corresponding tables.

4.1. TYPE 1 (INDEPENDENT OR SYNERGIC ELEMENTS)

This type can be either $\delta_1 = \delta_2 = 1$ (independent elements) or $\delta_1 > 1$, $\delta_2 \geq 1$ (synergic elements) and is illustrated in Figure 2.

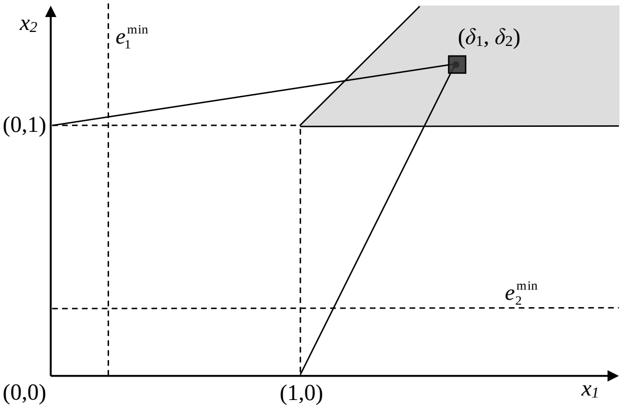

Fig. 2. *$n = 2$, case 1 (independent or synergic elements)*

The set of effects is represented by the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2). The feasible Pareto optimal boundary is made up of the single point δ. The input condition (2) guarantees the existence of the solution, given in Table 1.

Table 1. *The optimal solution in type 1*

	values
optimal effects	$x^* = (\delta_1, \delta_2)$
optimal quantities	$q_1 = 1, q_2 = 1$

4.2. TYPE 2 (PARTIALLY SYNERGIC AND PARTIALLY ANTAGONISTIC ELEMENTS)

This is the case $\delta_1 + \delta_2 > 1$, $\delta_1 \geq 1$, $\delta_2 < 1$. It is illustrated in Figure 3.

The set of effects is described by the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2).

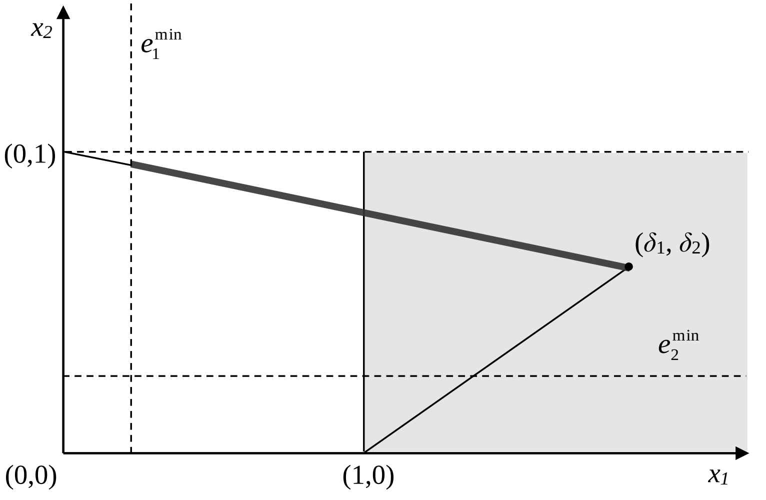

Fig. 3. $n = 2$, *case 2 (partially synergic and partially antagonistic elements)*

In order to simplify the notations, we define:

$$a_1 = \max(0,\ e_1^{\min})$$

$$b_1 = \min\left(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1) \right)$$

The existence of a solution requires, besides (2), the additional condition:

$$e_1^{\min} \leq b_1$$

This condition results in $a_1 \leq b_1$ and not-emptiness of the feasible Pareto optimal boundary. This boundary is the set of points (x_1, x_2) such that

$$x_1 \in [a_1, b_1]$$

$$x_2 = \frac{\delta_2 - 1}{\delta_1} x_1 + 1$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$and/or $e_2^{\min}$as follows:

- by fixing $e_2^{\min}$, we can use $e_1^{\min} = \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)$;
- by fixing $e_1^{\min}$, we can use $e_2^{\min} = \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1$.

Other ways are also open, if both $e_1^{\min}$ and $e_2^{\min}$are modified. The solution is given in the final row of Table 2.

Table 2. *The optimal solution in type 2*

existence condition	$e_1^{\min} \leq \min\left(\delta_1, \dfrac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)\right)$	
extremes of the feasible P.O. boundary	$\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \; \dfrac{\delta_2 - 1}{\delta_1} e_1^{\min} \; + \; 1\right)$ $\mathrm{R} = (R_1, R_2) = \left(\dfrac{\delta_1}{\delta_2 - 1}(\max\left(\delta_2, e_2^{\min}\right) - 1), \; \max\left(\delta_2, e_2^{\min}\right)\right)$	
optimal effects	$L_2/L_1 \leq r \leq R_2/R_1$	$x^* = (w_1, \; w_2)$ $w_1 = \delta_1/(r\delta_1 - \delta_2 + 1)$ $w_2 = rw_1$
	$r > L_2/L_1$	$x^* = L$
	$r < R_2/R_1$	$x^* = R$
optimal solution	$L_2/L_1 \leq r \leq R_2/R_1$	$q_1^* = 1/(r\delta_1 - \delta_2 + 1)$ $q_2^* = 1$
	$r > L_2/L_1$	$q_1^* = e_1^{\min}/\delta_1$ $q_2^* = 1$
	$r < R_2/R_1$	$q_1^* = \dfrac{\max\left(\delta_2, e_2^{\min}\right) - 1}{\delta_2 - 1}$ $q_2^* = 1$

4.3. TYPE 3 (WEAKLY ANTAGONISTIC ELEMENTS)

With this type, we have $\delta_1 + \delta_2 \geq 1$, $\delta_1 < 1$, $\delta_2 < 1$. This is illustrated in Figure 4.

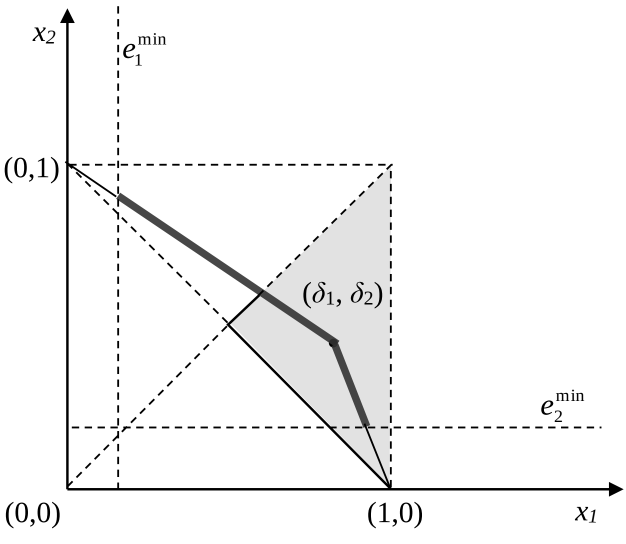

Fig. 4. $n = 2$, *case 3 (weakly antagonist elements)*

The set of effects is represented by the quadrangle having vertices (0, 0), (0, 1), (1, 0) and (δ_1, δ_2).

In order to simplify the notations, we define:

$a_1 = \max(0, e_1^{\min})$,
$b_1 = \min(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1))$
$a_2 = \max(\delta_1, e_1^{\min})$,
$b_2 = \min(1, \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1)$

The existence of a solution requires, besides (2), the additional condition:

$$e_1^{\min} \leq \max(b_1, b_2)$$

This condition results in $a_1 \leq b_1$ e $a_2 \leq b_2$ and the feasible Pareto optimal boundary is not empty. This boundary is the set of points (x_1, x_2) given by $R_1 \bigcup R_2$, where:

$$R_1 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{(\delta_2 - 1)}{\delta_1} x_1 + 1 \\ x_1 \in [a_1, b_1] \end{array} \right\} & if\ e_1^{\min} \leq \delta_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$R_2 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{\delta_2}{(\delta_1 - 1)} (x_1 - 1) \\ x_1 \in [a_2, b_2] \end{array} \right\} & if\ e_2^{\min} \leq \delta_2 \\ \emptyset & \text{otherwise} \end{cases}$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ as follows:

- by fixing $e_2^{\min}$, we can use

$$e_1^{\min} = \max\left(\frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1), \frac{\delta_1 - 1}{\delta_2} e_2^{\min} + 1\right);$$

- by fixing $e_1^{\min}$, we can use

$$e_2^{\min} = \min\left(\frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1, \frac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1)\right);$$

Other ways are also open, if both $e_1^{\min}$ and $e_2^{\min}$ are modified. The solution is given in the final row of Table 3.

Table 3. *The optimal solution in type 3*

existence condition	$e_1^{\min} \le \max\left(\min\left(\delta_1, \frac{\delta_1}{(\delta_2-1)}(e_2^{\min}-1)\right), \min\left(1, \frac{(\delta_1-1)}{\delta_2}e_2^{\min}+1\right)\right)$	
extremes of the feasible P.O. boundary	$\mathrm{L} = (L_1, L_2) = \begin{pmatrix} e_1^{\min}, & \left(\frac{\delta_2-1}{\delta_1}e_1^{\min}+1\right)\chi\left(e_1^{\min} \le \delta_1\right) + \\ & +\left(\frac{\delta_2}{\delta_1-1}(e_1^{\min}-1)\right)\chi\left(e_1^{\min} > \delta_1\right) \end{pmatrix}$ $\mathrm{R} = (R_1, R_2) = \begin{pmatrix} \left(\frac{\delta_1-1}{\delta_2}e_2^{\min}+1\right)\chi\left(e_2^{\min} \le \delta_2\right) + & \\ +\left(\frac{\delta_1}{\delta_2-1}(e_2^{\min}-1)\right)\chi\left(e_2^{\min} > \delta_2\right), & e_2^{\min} \end{pmatrix}$	
optimal effects	$r > L_2/L_1$	$x^* = L$
	$r < R_2/R_1$	$x^* = R$
	$\delta_2/\delta_1 \le r \le L_2/L_1$	$x^* = (w_1, w_2)$ $w_1 = \delta_1/(r\delta_1 - \delta_2 + 1)$ $w_2 = rw_1$
	$R_2/R_1 \le r \le \delta_2/\delta_1$	$x^* = (w_1, w_2)$ $w_1 = -\delta_2/(r\delta_1 - r - \delta_2)$ $w_2 = rw_1$
optimal quantities	$r > L_2/L_1$	$q_1^* = \frac{e_1^{\min}}{\delta_1}\chi\left(e_1^{\min} \le \delta_1\right) + \chi\left(e_1^{\min} > \delta_1\right)\ q_2^* =$ $= \chi\left(e_1^{\min} \le \delta_1\right) + \frac{e_1^{\min}-1}{\delta_1-1}\chi\left(e_1^{\min} > \delta_1\right)$
	$r < R_2/R_1$	$q_1^* = 1$ $q_2^* = \frac{e_2^{\min}}{\delta_2}\chi\left(e_2^{\min} \le \delta_2\right) + \frac{e_2^{\min}}{1-\delta_1}\chi\left(e_2^{\min} > \delta_2\right)$
	$\delta_2/\delta_1 \le r \le L_2/L_1$	$q_1^* = \dfrac{1}{r\delta_1 + 1 - \delta_2}$ $q_2^* = 1$
	$R_2/R_1 \le r \le \delta_2/\delta_1$	$q_1^* = 1$ $q_2^* = -\dfrac{r}{r\delta_1 - \delta_2 - r}$

4.4. TYPE 4 (STRONGLY ANTAGONISTIC ELEMENTS)

This is the case $\delta_1 + \delta_2 < 1$. This is illustrated in Figure 5.

It may be deduced from Carfì (2009e, pages 42–44) that the set of effects is the pseudo-triangle with vertices (0, 0), (0, 1), and (1, 0), delimited at North-East by the curve now to be defined. Having called $\delta_1^{'} = 1 - \delta_1$ and $\delta_2^{'} = 1 - \delta_2$, the resulting line is the union of:

- the segment of extremes (0, 1) and $H = (H_1,\ H_2) = (\delta_1^2/\delta_2^{'}, \delta_1^{'})$,
- the segment of extremes (1, 0) and $K = (K_1,\ placeK_2) = (\delta_2^{'}, \delta_2^2/\delta_1^{'})$,
- the section of the curve between H and K, having equation $x_2 = (1 - \sqrt{\delta_2^{'} x_1})^2/\delta_1^{'}$

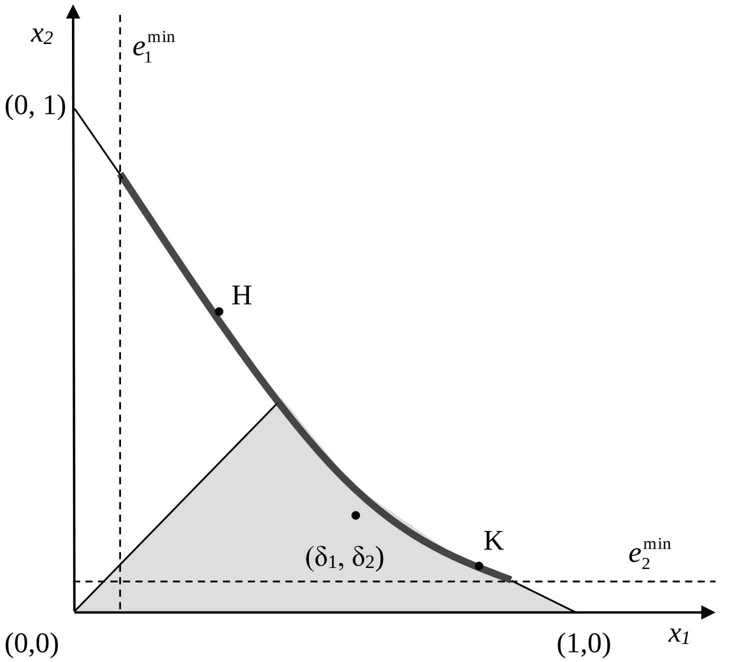

Fig. 5. $n = 2$, *case 4 (strongly antagonist elements)*

Remark. For other examples of similar calculations, we suggest to read the papers by Carfì and Schilirò (2014a, 2014b, 2013, 2012a, 2012b, 2012c, 2012d, 2011a, 2011b, 2011c) and by Carfì (2012a, 2012b, 2010, 2009a, 2009b, 2009c, 2009d, 2009e, 2008); the interested readers could also see Carfì and Musolino (2015a, 2015b, 2014a, 2014b, 2013a, 2013b, 2013c, 2012a, 2012b, 2012c, 2011a, 2011b). Other important applications, of the complete examination methodology, are shown in Agreste, Carfì, and Ricciardello (2012), Arthanari, Carfì and Musolino (2015), Baglieri, Carfì, and Dagnino (2012), Carfì and Fici (2012), Carfì, Gambarelli and Uristani (2013), Carfì and Lanzafame (2013), Carfì, Patanè, and Pellegrino (2011), Carfì and Romeo (2015).

Note that H belongs to the segment connecting $(0, 1)$ and (δ_1, δ_2), and K belongs to the segment connecting $(1, 0)$ and (δ_1, δ_2); then $H_1 \leq \delta_1$ and $H_2 \leq \delta_2$.

In order to simplify the notations, we define:

$$a_1 = \max(0, e_1^{\min}),$$
$$b_1 = \min\Big(H_1,\ \tfrac{\delta_1}{(\delta_2-1)}(e_2^{\min} - 1)\Big)$$
$$a_2 = \max(K_1, e_1^{\min}),$$
$$b_2 = \min\Big(1, \tfrac{(\delta_1-1)}{\delta_2} e_2^{\min} + 1\Big)$$
$$a_3 = \max(H_1, e_1^{\min}),$$
$$b_3 = \min\Big(K_1, \tfrac{\left(1-\sqrt{(1-\delta_1)e_2^{\min}}\right)^2}{1-\delta_2}\Big)$$

The existence of a solution requires, besides (2), the additional condition

$$e_1^{\min} \leq \max(b_1, b_2, b_3)$$

This condition results in $a_1 \leq b_1$, $a_2 \leq b_2$, and $a_3 \leq b_3$. In this case, the feasible Pareto optimal boundary is not empty. This boundary is the set of points (x_1, x_2) given by $R_1 \bigcup R_2 \bigcup R_3$, where:

$$R_1 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{(\delta_2 - 1)}{\delta_1} x_1 + 1 \\ x_1 \in [a_1, b_1] \end{array} \right\} & if\ e_1^{\min} \leq H_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$R_2 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{\delta_2}{(\delta_1 - 1)} (x_1 - 1) \\ x_1 \in [a_2, b_2] \end{array} \right\} & if\ e_2^{\min} \leq K_2 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$R_3 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{\left(1 - \sqrt{(1 - \delta_2) x_1}\right)^2}{1 - \delta_1} \\ x_1 \in [a_3, b_3] \end{array} \right\} & \begin{array}{l} \text{if } K_2 \leq e_2^{\min} \leq H_2 \\ \text{and } H_1 \leq e_1^{\min} \leq K_1 \end{array} \\ \emptyset & \text{otherwise} \end{cases}$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ in a way analogous to the previous cases:

- by fixing $e_2^{\min}$, we can use

$$e_1^{\min} = \max\left(\frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1), \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1, \frac{(1 - \sqrt{(1 - \delta_1) e_2^{\min}})^2}{1 - \delta_2} \right);$$

- by fixing $e_1^{\min}$, we can use

$$e_2^{\min} = \min\left(\frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1, \frac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1), \frac{(1 - \sqrt{(1 - \delta_2) e_1^{\min}})^2}{1 - \delta_1} \right);$$

Intermediate solutions are also possible, in which both $e_i^{\min}$ are modified. The solution is given in the final row of Table 4.

Table 4. *The optimal solution in type 4*

existence condition	$e_1^{\min} \le \max\left(\min\left(H_1,\ \frac{\delta_1}{(\delta_2-1)}(e_2^{\min}-1)\right),\ \min\left(1,\ \frac{(\delta_1-1)}{\delta_2}e_2^{\min}+1\right), \min\left(K_1,\ \frac{\left(1-\sqrt{(1-\delta_1)e_2^{\min}}\right)^2}{1-\delta_2}\right)\right)$	
extremes of the feasible P.O. boundary	$L=(L_1,L_2)=\left(e_1^{\min},\ \left(\frac{\delta_2-1}{\delta_1}e_1^{\min}+1\right)\chi\left(e_1^{\min}\le H_1\right) + \left(\frac{\delta_2}{\delta_1-1}(e_1^{\min}-1)\right)\chi\left(e_1^{\min}\ge K_1\right) + \left(\frac{(1-\sqrt{(1-\delta_2)e_1^{\min}})^2}{1-\delta_1}\right)\chi\left(K_1<e_1^{\min}<H_1\right)\right)$ $R=(R_1,R_2)=\left(\left(\frac{\delta_1-1}{\delta_2}e_2^{\min}+1\right)\chi\left(e_2^{\min}\le K_2\right) + \left(\frac{\delta_1}{\delta_2-1}(e_2^{\min}-1)\right)\chi\left(e_2^{\min}\ge H_2\right) + \frac{\left(1-\sqrt{(1-\delta_1)e_2^{\min}}\right)^2}{1-\delta_2}\chi\left(K_2<e_2^{\min}<H_2\right),\ e_2^{\min}\right)$	
optimal effects	$r \ge L_2/L_1$	$x^* = L$
	$r \le R_2/R_1$	$x^* = R$
	$r \ge H_2/H_1$ $r < L_2/L_1$ $r > R_2/R_1$	$x^* = (w_1, w_2)$ $w_1 = \delta_1/(r\delta_1 - \delta_2 + 1)$ $w_2 = rw_1$
	$H_2/H_1 \le r \le K_2/K_1$ $r < L_2/L_1$ $r > R_2/R_1$	$x^* = (w_1, w_2)$ $w_1 = \left(\frac{2((1-\delta_2)+r(1-\delta_1))-2\sqrt{\xi}}{2((1-\delta_2)+r(1-\delta_1))^2}\right)$ $w_2 = rw_1$ where $\xi = \sqrt{r(\delta_1-1)(\delta_2-1)}$
	$r \le K_2/K_1$ $r < L_2/L_1$ $r > R_2/R_1$	$x^* = (w_1,\ w_2)$ $w_1 = \left(\frac{\delta_2}{\delta_2+r(1-\delta_1)}\right)$ $w_2 = rw_1$
optimal quantities	$r \ge L_2/L_1$	$q_1^* = \left(\frac{e_1^{\min}}{\delta_1}\right)\chi\left(e_1^{\min}\le H_1\right) + \chi\left(e_1^{\min}\ge K_1\right) + \left(\frac{(e_1^{\min}(\delta_2-1)+\eta}{\eta(\delta_1-1)}\right)\chi\left(H_1<e_1^{\min}<K_1\right)$

Table 4. *cont.*

optimal quantities	$r \geq L_2/L_1$	$q_2^* = \chi\left(e_1^{\min} \leq H_1\right) + \left(\frac{e_1^{\min}-1}{\delta_1 - 1}\right)\chi\left(e_1^{\min} \geq K_1\right)$ $+ \left(\frac{\eta}{1-\delta_2}\right)\chi\left(H_1 < e_1^{\min} < K_1\right)$ where $\eta = \sqrt{e_1^{\min}(1-\delta_2)}$
	$r \leq R_2/R_1$	$q_1^* = \chi\left(e_2^{\min} \leq K_2\right) + \left(\frac{e_2^{\min}-1}{\delta_2 - 1}\right)\chi\left(e_2^{\min} \geq H_2\right) +$ $- \left(\frac{\theta + e_2^{\min}(\delta_1 - 1)}{\theta(\delta_2 - 1)}\right)\chi\left(K_2 < e_2^{\min} < H_2\right)$ $q_2^* = \left(\frac{e_2^{\min}}{\delta_2}\right)\chi\left(e_2^{\min} \leq K_2\right) + \chi\left(e_2^{\min} \geq H_2\right) +$ $+ \left(\frac{\theta}{1-\delta_1}\right)\chi\left(K_2 < e_2^{\min} < H_2\right)$ where $\theta = \sqrt{e_2^{\min}(1-\delta_1)}$
	$r \geq H_2/H_1$ $r < L_2/L_1$ $r > R_2/R_1$	$q_1^* = -\frac{\delta_1 - 1}{2(\delta_2 - 1)^2\sqrt{(\delta_1-1)/(\delta_2-1)}}$ $q_2^* = -\frac{\delta_2 - 1}{2(\delta_1 - 1)^2\sqrt{(\delta_2-1)/(\delta_1-1)}}$
	$H_2/H_1 \leq r \leq K_2/K_1$ $r < L_2/L_1$ $r > R_2/R_1$	If $\delta_1 = \delta_2$ $q_1^* = -\left(\frac{1}{2}\frac{1}{\sqrt{(\delta_2-1)/(\delta_1-1)}}\frac{\delta_2 - 1}{(\delta_1-1)^2}\right)$ $q_2^* = -\left(\frac{1}{2}\frac{1}{\sqrt{(\delta_1-1)/(\delta_2-1)}}\frac{\delta_1 - 1}{(\delta_2-1)^2}\right)$ otherwise $q_1^* = -\left(\frac{\delta_1 - 1 + \xi}{(\delta_1-1)(\delta_1-\delta_2)}\right)$ $q_2^* = -\left(\frac{\delta_2 - 1 + \xi}{(\delta_2-1)(\delta_1-\delta_2)}\right)$ where $\xi = \sqrt{(\delta_1-1)(\delta_2-1)}$
	$r \leq K_2/K_1$ $r < L_2/L_1$ $r > R_2/R_1$	$q_1^* = 1$ $q_2^* = -\frac{r}{r\delta_1 - \delta_2 - r}$

5. TRUNCATED BILINEAR CASE

These cases involve situations in which the effects (beyond a certain maximum level) fall to zero. The symbol χ will be used in the text to denote the indicator function; i.e.,

$$\chi\,(\text{condition}) = \begin{cases} 1 & \text{if the condition is satisfied} \\ 0 & \text{if the condition is not satisfied} \end{cases}$$

Using the above symbol, we can define the effect-function $e(q)$ of truncated bilinear cases as follows:

$$e_1 = \chi(q_1(1 - q_2) + q_1 q_2 \delta_1 \leq 1)[q_1(1 - q_2) + q_1 q_2 \delta_1]$$

$$e_2 = \chi(q_2(1 - q_1) + q_1 q_2 \delta_2 \leq 1)[(1 - q_1)q_2 + q_1 q_2 \delta_2]$$

5.1. TYPE 1 TRUNCATED (INDEPENDENT OR SYNERGIC ELEMENTS)

This type corresponds either to ($\delta_1 = \delta_2 = 1$) or ($\delta_1 > 1$, $\delta_2 \geq 1$). This is illustrated in Figure 6.

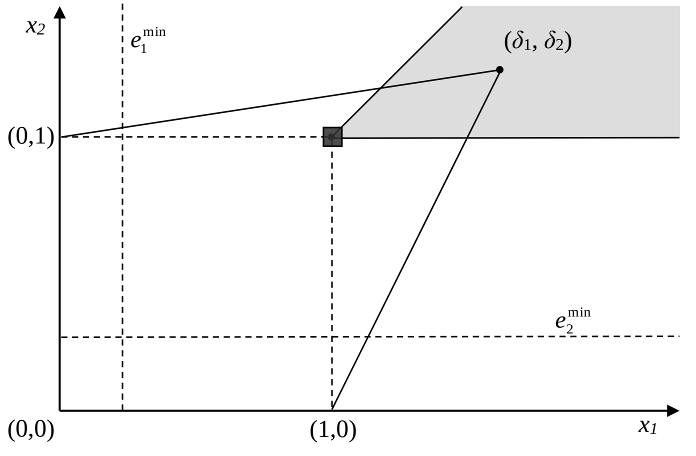

Fig. 6. $n = 2$, *case 1 (independent or synergic elements)*

The set of effects is the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2). The feasible Pareto optimal boundary is made up of the single point (1, 1). Therefore, $x_1 = x_2 = 1$.

The input condition (2) guarantees the existence of the solution, which is given in Table 5.

Table 5. *the optimal solution in type 1T*

	$\delta_1 = \delta_2 = 1$	$\delta_1 > 1$ $\delta_2 = 1$	otherwise
optimal effects	$x^* = (1,1)$	$x^* = (1,1)$	$x^* = (1,1)$
optimal quantities	$q_1 = \dfrac{1}{\delta_1}$ $q_2 = 1$	$q_1 = 1$ $q_2 = 1$	$q_1 = \dfrac{1}{1 + q_2(\delta_2 - 1)}$ $q_2 = \dfrac{\sqrt{\kappa^2 - \kappa + 4(\delta_1 - 1)}}{2(\delta_1 - 1)}$ $\kappa = (1 - (\delta_1 - 1) + (\delta_2 - 1))$

5.2. TYPE 2 TRUNCATED (PARTIALLY SYNERGIC AND PARTIALLY ANTAGONISTIC ELEMENTS)

This is the case $\delta_1 + \delta_2 > 1$, $\delta_1 \geq 1$, $\delta_2 < 1$. This is illustrated in Figure 7.

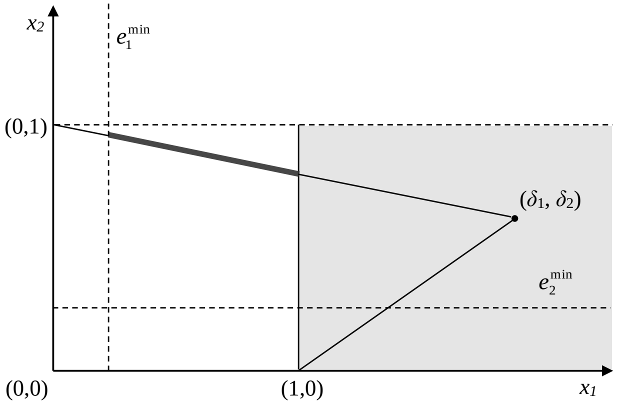

Fig. 7. *$n = 2$, case 2 (partially synergic and partially antagonistic elements)*

The set of effects is the quadrangle having vertices (0, 0), (0, 1), (1, 0), and (δ_1, δ_2). Although it is analogous to Type 2 in the case given in the previous paragraph, the effects cannot exceed the value of 1 in this case.

In order to simplify the notation, we define:

$$a_1 = \max\left(0, e_1^{\min}\right)$$
$$b_1 = \min\left(1, \frac{\delta_1}{\delta_2 - 1}\left(e_2^{\min} - 1\right)\right)$$

Using the above notations, the conditions for the existence of a solution, calculations, and all related considerations are the same as those for Section 4.2. The solution is given in the final row of Table 6.

Table 6. *The optimal solution in type 2T*

existence condition	$e_1^{\min} \leq \min\left(1, \frac{\delta_1}{\delta_2 - 1}\left(e_2^{\min} - 1\right)\right)$	
extremes of the feasible P.O. boundary	$\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1\right)$ $\mathrm{R} = \left(\frac{\delta_1}{\delta_2 - 1}\left(\max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right) - 1\right), \max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right)\right)$	
optimal effects	$L_2/L_1 \leq r \leq R_2/R_1$	$x^* = (w_1, w_2)$ $w_1 = \delta_1/(r\delta_1 - \delta + 1)$ $w_2 = rw_1$

Table 6. *cont.*

	$r > L_2/L_1$		$x^* = L$
	$r < R_2/R_1$		$x^* = R$
optimal solution	$L_2/L_1 \leq r \leq R_2/R_1$		$q_1^* = 1/(r\delta_1 - \delta_2 + 1)$ $q_2^* = 1$
	$r > L_2/L_1$		$q_1^* = e_1^{\min}/\delta_1$ $q_2^* = 1$
	$r < R_2/R_1$	$\delta_1 = 1$	$q_1^* = R_1$ $q_2^* = 1$
		$\delta_1 > 1$	$q_1^* = \dfrac{\frac{\delta_1}{\delta_2 - 1}\max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right) - 1}{1 + q_2(\delta_1 - 1)}$ $q_2^* = \dfrac{(\delta_1 - \vartheta - 1) + \sqrt{(\delta_1 - \vartheta - 1)^2 + 4\vartheta(\delta_1 - 1)}}{2(\delta_1 - 1)}$ $\vartheta = \max\left(\dfrac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right)$

5.3. TYPES 3 AND 4 TRUNCATED

Types 3 and 4 truncated are the same as those of the bilinear free case. We therefore refer the reader to the considerations given in Sections 4.3 and 4.4.

6. AN ALGORITHM

The input data is $\delta, e^{\min}$, and the option free-truncated function.

We begin by acquiring the data and by doublechecking the conditions required in Section 2.

With regard to r, it is quite possible that the user is unable to determine this *a priori*, and it is therefore useful to supply the user with an interval of variability *r_ int* to allow this parameter to be established.

The algorithm proceeds using the tables given in Sections 4 and 5. If a feasible solution is reached, the process stops. Otherwise, the user has to be informed that ${e_1}^{\min}$ and/or ${e_2}^{\min}$ are too binding and should be modified, giving suitable indications for doing this.

A definitive calculation can now be made and the results communicated.

7. SOME APPLICATIONS

In Industrial Economics, finding the optimal quantities of goods to be produced is a well-known problem. Some goods may be complementary or substitutes; hence, their demands may influence each other. If the same firm produces such kinds of goods, it is profitable to optimally decide the production quantities of each product. This decision also depends on the willingness of the decision-maker to potentially sacrifice part of the demand of one product. This willingness to cannibalize a product depends on various factors, examples being the future market situation of the two products and a company's desire to place itself at a strategic advantage in an emerging market (for a detailed analysis of the factors influencing the willingness to cannibalize, see Chandy *et al.*, 1998; Nijssen *et al.*, 2004 and Battaggion *et al.*, 2009).

The model can be used analogously in Public Economics to calibrate two differing economic policies that are interfering with each other.

In Medicine and Veterinarian practice, the balance of interfering drugs is usually performed by successive approximations, keeping the patient monitored.

Finally, further applications can be seen in Zootechnics (to optimize diets), in Agriculture (to calculate dosages of parasiticides or additives so as to increase production), and so on.

8. SOME OPEN PROBLEMS

Figure 8 shows a graph corresponding to Figure 1 for the case $n = 3$. Working with graphic methods (as in this paper) is more difficult in the case of multilinear functions, but not impossible.

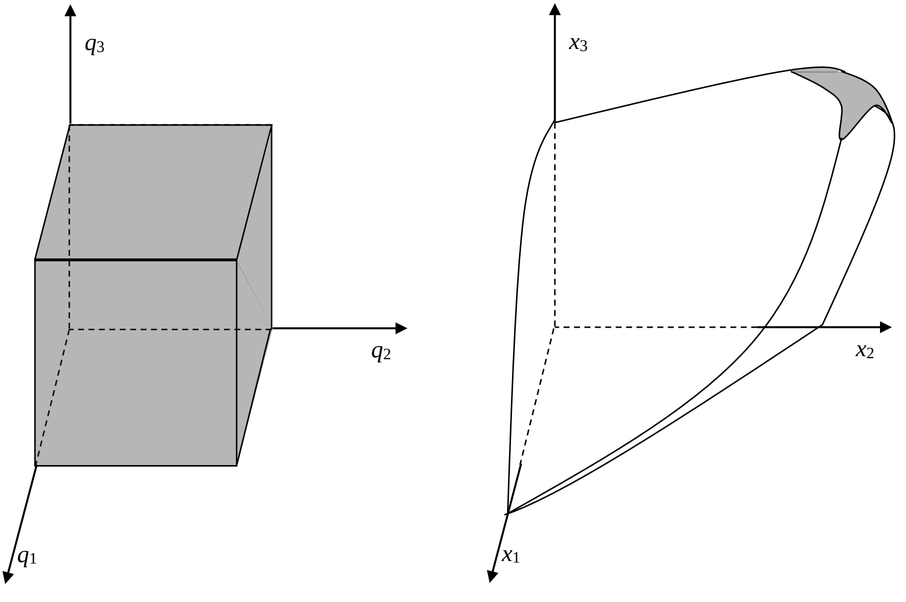

Fig. 8. $n = 3$

Further studies could apply this technique to Cooperative Game Theory, where bilinear functions are often applied (see Fragnelli and Gambarelli, 2013a, 2013b).

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Allocating Pooled Inventory According to Contributions and Entitlements

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Abstract. Inventory pooling, whether by centralization of stock or by mutual assistance, is known to be beneficial when demands are uncertain. But when the retailers are independent, the question is how to divide the benefits of pooling. e consider a decentralized inventory-pooling scheme where a retailer's entitlements to the allocation during a shortage depend on his/her contributions to the pool. We derive the Nash equilibrium and specialize it to symmetric cases.

Keywords: inventory pooling; retailer contributions; entitlements; Nash equilibrium

Mathematics Subject Classification: 90, 91

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1. INTRODUCTION

Inventory pooling in a decentralized system raises the issue of how to divide benefits among the parties, which is typically done using concepts from cooperative games (e.g., Gerchak and Gupta 1991; Hartman *et al.*, 2000; Müller *et al.*, 2002; Montrucchio *et al.*, 2012); In particular, if the before-pooling optimal expected profits are $\pi_1^*(q_1^*)$and $\pi_2^*(q_2^*)$, and the optimal expected profit after pooling is $\pi^*(q^*)$, the gain, $\pi^*(q^*) - \pi_1^*(q_1^*) - \pi_2^*(q_2^*)$ could be divided proportionally to $\pi_1^*(q_1^*)$ and $\pi_2^*(q_2^*)$. That allocation can be shown to belong to the (non-empty) core. We propose a very different "operational" *non-cooperative* scheme according to which the parties have entitlements that are increasing functions of their *contributions* to the pool rather than a typical division of benefits. The idea was proposed by Ben-Zvi and Gerchak (2012, sec. 11.5); but here, we analyze the consequences of a somewhat-modified scheme. The scheme can be shown to be beneficial to all parties vis-á-vis a no-pooling situation, namely is in core.

The scheme works as follows: each party (e.g., retailer) contributes a quantity of its choice to the pool. There is no inventory at the retailers' locales. The choice of these quantities is simultaneous. If the quantity of pooled inventory is not sufficient to

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meet all realized demands, a retailer whose contribution exceeds its demand receives an allocation equal to its demand. The other retailer (whose demand exceeds its contribution) receives the rest of the inventory. If both demands exceed the respective contributions, each retailer receives its contribution. The scheme can be shown to be beneficial for all parties vis-ŕ-vis a no-pooling scenario. Mathematically (but not in type of motivations), our model is related to Shao *et al.* (2011).

The simultaneous selection of quantities to contribute to the pool is a non-cooperative game. In a symmetric scenario (unit cost, revenue, demand distributions), we solve directly for the (common) order quantity. We provide an example.

2. GENERAL MODEL

Let r_i be the unit revenue and c_i unit production cost of party i, $r_i > c_i \forall i$. The distributions of the independent demands, X_1 and X_2, are F_1 and F_2. The parties are simultaneously looking for their best contribution quantities q_1 and q_2. The actual allocations to the parties as a function of demand realizations $(x_1, x_2)$are denoted by a_1 and a_2.

The policy (contract) is as follows.

If $x_i \leq q_i$, then $a_i = x_i$;

if, in addition, $x_j \leq q_j$ then $a_j = x_j$;

if $x_j > q_j$ and $x_i + x_j \leq q_i + q_j$, then $a_j = x_j$;

if $x_j > q_j$ and $x_i + x_j > q_i + q_j$, then $a_j = q_i + q_j - x_i \, (\leq x_j)$. [This is the case where inventory pooling helps party j].

If $x_i > q_i$ and $x_j > q_j$, then $a_i = q_i$ and $a_j = q_j$. ‖

So party $i's$ expected profit is:

$$
\begin{aligned}
E_i = -c_i q_i + r_i \Bigg\{ & \int\limits_{x_i=0}^{q_i} x_i \int\limits_{x_j=0}^{q_j} f_i(x_i) f_j(x_j) \, dx_j dx_i \\
& + \int\limits_{x_i=q_i}^{q_i+q_j} \int\limits_{x_j=0}^{q_i+q_j-x_i} x_i f_i(x_i) f_j(x_j) \, dx_j dx_i \\
& + \int\limits_{x_i=q_i}^{q_i+q_j} \int\limits_{x_j=q_i+q_j-x_i}^{q_j} (q_i + q_j - x_j) f_j(x_j) f_i(x_i) \, dx_j dx_i \\
& + \int\limits_{x_i=q_i+q_j}^{\infty} \int\limits_{x_j=0}^{q_j} (q_i + q_j - x_j) f_j(x_j) f_i(x_i) \, dx_j dx_i \\
& + \int\limits_{x_i=q_i}^{\infty} \int\limits_{x_j=q_j}^{\infty} q_i f_i(x_i) f_j(x_j) \, dx_j dx_i \Bigg\}
\end{aligned}
$$

so:

$$dE_i/dq_i = -c_i + r_i \Big\{ 2\overline{F}_j(q_j)\overline{F}_i(q_i) - q_i f_i(q_i)\overline{F}_j(q_j) + q_j F_j(q_j) - q_i f_j(q_j) F_i(q_i+q_j) + \int_{x_i=q_j}^{q_i+q_j} F_j(q_i+q_j-x_i)\,dx_i + F_j(q_j)\overline{F}_i(q_i+q_j) - q_i f_j(q_j) \Big\}$$

Note that E_i is an explicit function of only c_i and r_i. On the other hand, it depends on both F_i and F_j.

In the symmetric case ($c_1 = c_2 \equiv c,\ \ r_1 = r_2 \equiv r$), $F_1 = F_2 \equiv F$, where we will look for a symmetric equilibrium, $q_1 = q_2 \equiv q$, we have ($r > c$)

$$dE/dq = -c + r\Big\{ 2\left[\overline{F}(q)\right]^2 - qf(q)\overline{F}(q) + qF(q) - qf(q)F(2q) + F(q)\overline{F}(2q) - qf(q) + \int_q^{2q} F(2q-x)dx \Big\}$$

$$dE/dq|_{q=0} = -c + 2r > 0 \Rightarrow q^* > 0$$

3. UNIFORMLY DISTRIBUTED DEMANDS

Here $X \sim U[0,1]$, so $0 \le q \le 1$.

If $q \le \frac{1}{2}$, the optimality condition becomes:

$$-c + r\left(-3q^2 - 3q + 2\right) = 0$$

i.e., $3q^2 + 6q - 4 + \frac{2c}{r} = 0$.

So:

$$q^* = 1 - \frac{\sqrt{21 - \frac{6c}{r}}}{3} \left(< \frac{1}{2}\right)$$

.

If $q > \frac{1}{2}$, the optimality condition becomes:

$$-c + r\left(\frac{9}{2}q^2 - 7q + 2\right) = 0$$

i.e., $q^2 - 14q + 4 - \frac{2c}{r} = 0$

$$\Rightarrow q^{**} = 7 - \sqrt{45 + \frac{2c}{r}}$$

However, $q^{**} < \frac{1}{2}$. Thus, the solution in this range is either boundary $q = \frac{1}{2}$ or $q = 1$.

4. CONCLUDING REMARKS

In partnerships, the partners' entitlements to profits are often based on their ownership shares. We use a similar philosophy in allocating scarce inventories. That translates to a non-cooperative game.

Issues that were not explored in this context but might be of interest include:

1) Non-Linear productions costs (Gerchak and Schwarz, 2014).
2) Holding some inventory at the retailers' locales ("partial pooling"), possibly with transshipment costs from the pool.
3) More insight into the case of non-symmetric parameters.

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On the Non-Symmetric Nash and Kalai–Smorodinsky Bargaining Solutions

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Abstract. Recently, in some negotiation application areas, the usual assumption that negotiators are symmetric has been relaxed. In particular, weights have been introduced to the Nash Bargaining Solution to reflect the different powers of the players. Yet, we feel that operating with non-symmetric bargaining solutions and their implications is not well understood. We analyze the properties and optimization of the non-symmetric Nash Bargaining Solution and of a non-symmetric Kalai–Smorodinsky Bargaining Solution. We provide extensive comparative statics, then comment on the implications of the concepts in supply chain coordination contexts.

Keywords: Nash Bargaining Solution; Non- Symmetric; Kalai–Smorodinsky Bargaining Solution, Supply Chain Coordination

Mathematics Subject Classification: 91A

JEL Classification: C71

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1. INTRODUCTION

The original Nash Bargaining Solution (NBS) is symmetric in the excesses of the players' utilities over their disagreement utilities. Thus, bargainers are envisioned to be on "equal footing." In an attempt to give one player a "priority" over the other(s), Kalai (1977) axiomatized and presented the non-symmetric NBS (NSNBS). It uses different powers of the excesses over the disagreement values, summing to one. While Kalai (1977) and others explored the axiomatics of the NSNBS, its optimization and economic implications have not been fully explored. As the NSNBS is being recently used in various supply chain settings (Nagarajan and Sosic, 2008; Wu *et al.*, 2009; Mantin *et al.*, 2014), it is important that the SCM and other communities that use such bargaining models will know their general properties. We also consider a lesser-known, non-symmetric version of the Kalai–Smorodinsky solution, where the excesses of the ideal point over the disagreement point are taken to differing powers (Durba, 2001). We then discuss the uses of these concepts in Supply Chain settings.

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2. NBS ANALYSIS

The non-symmetric, two-player NBS is the solution of the maximization problem:

$$\max_A f(A) \equiv (A-x)^{\alpha} \ (K-A-y)^{1-\alpha} \ , \ 0 \leq \alpha \leq 1, A \geq x, K \geq x+y,$$

where K is the quantity of the resource to be divided (the first player receives A) and x and y are the respective disagreement values. Note that for $A = x$ or $A = K - y$, $f = 0$. The parameter α reflects the relative priority (power) of the first player. When $\alpha = 0.5$, one obtains the symmetric NBS. Now:

$$\partial f/\partial A = (A-x)^{\alpha-1} \ (K-A-y)^{-\alpha} \left[\alpha \ (K-x-y) \ - \ (A-x)\right]$$

so:

$$\partial^2 f/\partial A^2 = \ (A-x)^{\ \alpha-2} \ (K-A-y)^{-\alpha-1} \Big\{\alpha \ (\alpha-1) \ (K-A-y)^2$$
$$+ \ \alpha^2 \ \left((K-x-y) \ (A-x) \ - \alpha \, (A-x)^2\right) - \alpha \ (A-x) \, (K-A-y)\Big\}$$

the sign of which is that of $(\alpha-1)\,(K-A-y)^2 - \alpha \ (A-x)$ which is negative since $0 \leq \alpha \leq 1$.

Thus, the function is concave in A.

Now:

$$\partial f/\partial A \ = 0 \Leftrightarrow \quad \text{the only relevant solution is } A^* \ = \ \alpha \, (K-y-x) \, + x$$

Note that A^* linearly increases inα, which is intuitive, as α signifies the power of the first player. It also linearly increases in x and linearly decreases in y. When a player has a large “fallback” position, he will have to be “compensated” more.

A more-general scenario is where if one player gets A, the other gets $g(A)$, where $g' < 0$ and $g'' < 0$. Thus, although what is left for the second player is decreasing in the first's use, that impact is decreasing. So, the objective becomes:

$$\max_A h(A) \equiv (A-x)^{\alpha} \ [g\,(A)-y]^{1-\alpha}$$

Note that $h\,(x) \ = h\left(g^{-1}(y)\right) = 0$; as with such allocation, both players will rather stay at their disagreement point.

Now:

$$dh/\partial A = \ (A-x)^{\alpha-1} \ [g\,(A)-y]^{-\alpha} \ \left\{\alpha\,[g\,(A) \ -y] + (1-\alpha)(A-x)g'(A)\right\}$$

so:

$$\partial^2 h/\partial A^2 \ = (\alpha-1) \ (A-x)^{\alpha-2} \ [g\,(A)-y]^{-\alpha-1} \Big\{-\alpha g'\,(A)\,[\alpha \ (g\,(A) \ -y)]$$
$$+ (1-\alpha) \ (A-x) \ g'\,(A) \ + (g\,(A) \ -y) \ [g'\,(A) + Ag''\,(A)]\Big\}$$

Since $0 \leq \alpha \leq 1$, the quantity in {} is positive, as $\left(1-\alpha^2\right) g'(A)\,[g(A)-y]$ is negative and so is $(1-\alpha)\, g'(A)\,(A-x)$.

Thus, h is concave in A.

The optimality condition is:

$$\alpha[g(A^*) - y] + (1-\alpha)(A^* - x)\, g'(A^*) \;=\; 0$$

The direction of the dependence of A^* on α can be obtained by comparative statics:

$$dA^*/d\alpha \;=\; \frac{-g(A) + y + [A-x]\, g'(A)}{(1-\alpha)\;(A-x)\; g''(A) \;+ g'(A)}$$

As both numerator and denominator are negative, $dA^*/d\alpha \geq 0$.

That is, the more powerful the first player, the more he is allocated.

3. EXAMPLES

3.1. EXAMPLE 1

$$A^2 + B^2 = K \;\Rightarrow\; g(A) \;=\; \sqrt{K - A^2}$$

$$g'(A) = -A\Big/\sqrt{K-A^2} < 0$$

$$g''(A) = K\Big/K(K-A^2)^{\frac{3}{2}} \leq 0 \Rightarrow \text{ concave}$$

$$0 \equiv \frac{\partial}{\partial A} = \alpha\left[\sqrt{K-A^2} - y\right] + (1-\alpha)\;(A-x)\,.\,\frac{-2A}{2\sqrt{K-A^2}}$$

so the optimality condition is:

$$\alpha K - \alpha y\sqrt{K - A^{*2}} + x(1-\alpha)A^* - A^{*2} = \;0$$

That leads to a quartic equation in A.

If $x = y = 0$, then $A^* = \sqrt{\alpha K}$.

If $y = 0$

$$A^{*2} - x(1-\alpha)\, A^* - \alpha K \;=\; 0$$

$$A^* = \frac{x(1-\alpha) \;+\; \sqrt{x^2(1-\alpha)^2 + 4\alpha K}}{2}$$

3.2. EXAMPLE 2

This example has a *convex* function g.

$$AB = K$$

$$A, B \geq 1 \Rightarrow g(A) = K/A \Rightarrow g'(A) = -K/A^2 \Rightarrow g''(A) = 2K/A^3 > 0$$

Thus:

$$\alpha[K/A - y] + (1-\alpha)(A-x)\left(-K/A^2\right) = 0$$

$$\Rightarrow A^* = \frac{K(2\alpha - 1) + \sqrt{4\alpha^2 K^2 - 4\alpha K^2 + K^2 + 4\alpha(1-\alpha)xyK}}{2\alpha y}$$

Easy to see that $\Delta \geq 0$, and that $A^* \geq 0$.

$\frac{dA^*}{d\alpha}$ has the sign of

$$2K\left[(2\alpha-1)(K-xy) + \sqrt{4\alpha^2 K^2 - 4\alpha K^2 + K^2 - 4\alpha(1-\alpha)xyK}\right]$$

Thus, if $\alpha < \frac{3-\sqrt{3}}{6} \approx 0.2$, the condition for $\frac{dA^*}{d\alpha} > 0$ is $K \geq \frac{xy(2\alpha-1)}{2(6\alpha^2-6\alpha+1)}$. If $\alpha > \frac{3-\sqrt{3}}{6}$, $\frac{dA^*}{d\alpha} < 0$ always. Interestingly, here an increase in one's power increases its share only if his power was initially low.

4. NBS WITH N PLAYERS

Here, the function being maximized is:

$$g(A_{1,\ldots,}A_{n-1}) = \left[\prod_{i=1}^{n-1}(A_i - x_i)^{\alpha_i}\right]\left(K - \sum_{i=1}^{n-1} A_i - x_n\right)^{\alpha_n}$$

$$\alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1.$$

This model was also discussed by Kalai (1977).

Now:

$$\begin{aligned}\frac{\partial g}{\partial A_i} &= \alpha_i (A_i - x_i)^{\alpha_i - 1} \prod_{j \neq i}(A_j - x_j)^{\alpha_j}\left(K - \sum_{i=1}^{n-1} A_i - x_n\right)^{\alpha_n} \\ &- \left[\prod_{i=1}^{n-1}(A_i - x_i)^{\alpha_i}\right]\alpha_n\left(K - \sum_{i=1}^{n-1} A_i - x_n\right)^{\alpha_n - 1} \\ &= \left[\prod_{j\neq i}^{n-1}(A_j - x_j)^{\alpha_j}\right](A_i - x_i)^{\alpha_i - 1}\left(K - \sum_{i=1}^{n-1} A_i - x_n\right)^{\alpha_n - 1} \\ &\times \left\{\alpha_i (A_i - x_i)^{\alpha_i - 1}\left(K - \sum_{i=1}^{n-1} A_i - x_n\right) - \alpha_n\right\} = 0\end{aligned}$$

As:

$$A_i > x_i \ \Rightarrow$$
$$\alpha_i \left(K - \sum_{i=1}^{n-1} A_i - x_n\right) (A_i - x_i)^{\alpha_i - 1} = \alpha_n \,. \Rightarrow$$
$$A_i^* = x_i + [\tfrac{\alpha_n}{\alpha_i}(K - \sum_{j=1}^{n-1} A_j - x_n]^{\frac{1}{\alpha_i - 1}}, \quad i = 1, \ldots.., n-1$$

Note that, since $\frac{1}{\alpha_i - 1} < -1$, A_i^*decreases in the quantity in [ ]. Thus,$A_i^* \uparrow x_i$, and, $A_i^* \downarrow K,\ A_i \downarrow \sum A_j$,$A_i^* \downarrow \alpha_n$, $A_i^* \uparrow \alpha_i$.

The non-intuitive implication is that A_i^* is decreasing in K. We cannot explain this implication.

5. K-S SOLUTION

Let $(d_1\,,\,d_2)$ be the disagreement point and $(M,\ N)$ the ideal point, where each player obtains his highest utility on S, where $N \geq d_2$, $M \geq d_1$. Then, the K-S solution is the pair (u_1, u_2), where $u_1 + u_2 = S$, satisfying

$$\frac{u_2 - d_2}{u_1 - d_1} = \frac{N - d_2}{M - d_1}, u_1 \geq d_1,\, u_2 \geq d_2$$

(though the utilities might be negative) (Kalai and Smorodinsky, 1975).

An asymmetric generalization could be (Dubra, 2001):

$$\frac{u_2 - d_2}{u_1 - d_1} = \frac{(N - d_2)^{\alpha}}{(M - d_1)^{1-\alpha}}\,,\ 0 \leq \alpha \leq 1.$$

It follows that:

$$u_2 \ = \ \frac{d_2\,(M - d_1)^{1-\alpha} + (S - d_1)\ (N - d_2)^{\alpha}}{(M - d_1)^{1-\alpha} + (N - d_2)^{\alpha}}$$

while:

$$u_1 = S - u_2.$$

One can also show that:

$$u_2'\,(\alpha) = \frac{(u_2 - d_2)\,(M - d_1)^{1-\alpha} \log{(M - d_1)} + (S - u_2 - d_1)\,(N - d_2)^{\alpha} \log{(N - d_2)}}{(M - d_1)^{1-\alpha} + (N - d_2)^{\alpha}}$$

So, $u_2'\,(\alpha) > 0$. This is intuitive.

Note that $u_1 = S - u_2$ is similar to the linear resource constraint $A + B = K$ in NBS. One could thus try the generalization suggested there ($u_1 = g\,(u_2,\, S)$), but we shall not do so here.

6. EXAMPLE

Suppose that $u_1^2 + u_2^2 = S$. Note that $d_i^2 \leq S, \quad i = 1, 2$. The utilities may be negative.

So $\frac{\sqrt{S-u_1^2}-d_2}{u_1-d_1} = \frac{(N-d_2)^\alpha}{(M-d_1)^{1-\alpha}} \equiv R$ [note that $\frac{dR}{d\alpha} > 0$]

$$\Rightarrow \left(1 + R^2\right) u_1^2 - 2R\left(Rd_1 - d_2\right) u_1 + d_1^2 R^2 - S - 2d_1 d_2 R + d_2^2 = 0$$

$$u_1 = \frac{R\left(Rd_1 - d_2\right) \pm \sqrt{S\left(1 + R^2\right) - \left(Rd_1 - d_2\right)^2}}{1 + R^2}$$

To have $\Delta \geq 0$, we require that:

$$S \geq \frac{\left(Rd_1 - d_2\right)^2}{1 + R^2}$$

We note that, if $N \geq (M - d_1)^{-2} + d_2$, then $R^2 \geq 1$, and it can be shown that then, for the positive roots, $du_1/dR \geq 0$.

7. POSSIBLE RELEVANCE TO SUPPLY CHAIN ISSUES

Horizontal supply chains are created when several retailers (at the same echelon of a supply chain that faces random demands) share inventories. This can be achieved either by pooling inventories in a central location or by sharing inventories when one retailer would have a shortage and the other an excess. Such pooling/sharing will result in savings (e.g., Eppen, 1979); but, if the retailers are independent firms, the question is how to divide the savings. This issue was explored using NBS by Hanany and Gerchak (2008). If some retailers are more powerful than others, introducing a-symmetry (NSNBS or NS K-S) would be natural.

Vertical decentralized supply chains [e.g., supplier(s)/manufacturer(s) and retailer(s)] were explored extensively in operations-management literature (e.g., Cachon, 2003). There, after finding a type of contract that will coordinate the supply chain (i.e., make it behave as an integrated chain), the issue that arises is how to divide the profits. Mantin *et al.* (2014) propose an NSNBS manufacturer-retailer bargaining model. Possibly, the NSK-S solution could also be employed, since each party's ideal point is obtaining all of the profit.

8. CONCLUDING REMARKS

The Symmetric NBS is a Non-Transferable Utility (NTU) concept; that is, at that solution, all players' situations improve vis-á-vis their disagreement points. No player needs to compensate another for that to happen (as in TU), which is rather attractive. This property is maintained by the NSNBS, and also holds for the NS K-S solution. We provide some insight into the optimization of these non-symmetric functionals, and extensive comparative statistics for them.

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Interval Methods for Computing Strong Nash Equilibria of Continuous Games

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Abstract. The problem of seeking strong Nash equilibria of a continuous game is considered. For some games, these points cannot be found analytically, only numerically. Interval methods provide us with an approach to rigorously verify the existence of equilibria in certain points. A proper algorithm is presented. We formulate and prove propositions, that give us features which have to be used by the algorithm (to the best knowledge of the authors, these propositions and properties are original). Parallelization of the algorithm is also considered, and numerical results are presented. As a particular example, we consider the game of "misanthropic individuals", a game, invented by the first author, that may have several strong Nash equilibria depending on the number of players. Our algorithm is able to localize and verify these equilibria.

Keywords: strong Nash equilibria, continuous games, interval computations, numerical game solving

Mathematics Subject Classification: 65G40, 65K99, 91A06, 91A35, 91B50

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1. INTRODUCTION

Game theory tries to predict decisions and/or advise the decision makers on how to behave in a situation when several players (sometimes called "agents") have to choose their behavior (strategy; the i-th player chooses the strategy $x^i \in X_i$) that will also influence the others. Usually, we assume that each player tends to minimize his cost function (or maximize his utility) $q_i(x^1, \ldots, x^n)$.

So, each of the decision makers solves the following problem:

$$\begin{aligned} & \min q_i(x^1, \ldots, x^n) \qquad (1) \\ & \text{s.t.} \\ & x^i \in X_i \end{aligned}$$

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What solution are they going to choose?

One of the oldest, most famous, and still widely-used concepts is the Nash equilibrium (Nash, 1950). It can be defined as a situation (an assignment of strategies to all players), when each player's strategy is optimal against those of the others.

Formally, the tuple $x^* = (x^{1*}, \ldots, x^{n*})$ is a Nash equilibrium, if:

$$(\forall i = 1, \ldots, n)\ (\forall x^i \in X_i)\ q_i(x^{1*}, \ldots, x^{i-1*}, x^i, x^{i+1*}, \ldots, x^{n*}) \geq q_i(x^{1*}, \ldots, x^{n*}) \quad (2)$$

We shall use a shorter notation, also: $\forall i = 1, \ldots, n \quad \forall x^i \quad q_i(x^{\setminus i*}, x^i) \geq q_i(x^{\setminus i*}, x^{i*})$.

Such points, however, have several drawbacks – both theoretical (rather strong assumptions about the players' knowledge and rationality) and practical (they can be Pareto-inefficient; i.e., it is possible to improve the outcome of one player without worsening the others' results (Miettinen, 1999)).

Hence, several "refinements" to the notion have been introduced, including the strong Nash equilibrium (SNE, for short), in particular; see (Aumann, 1959). For such points, not only none of the players can improve their performance by changing strategy, but also no *coalition* of players can improve the performance of all of its members by mutually deviating from the SNE. Formally:

$$(\forall I \subseteq \{1, \ldots, n\}) \quad (\forall x^I \in \times_{i \in I} X_i) \quad (\exists i \in I) \quad q_i(x^{\setminus I*}, x^I) \geq q_i(x^{\setminus I*}, x^{I*}) \qquad (3)$$

Also, the notion of a k-SNE (or k-equilibrium) is sometimes encountered. Its definition is similar to ordinary SNE, but the coalition I in (3) can consist of k members at most. Obviously, a $(k + l)$-SNE is also a k-SNE (if $l > 0$) and, in particular, a SNE is also a k-SNE for any $k = 1, 2, \ldots, n$.

Strong Nash equilibria have been long thought to be too restrictive to be useful in practical situations, but they have received increased interest in recent years. Apparently, there exist some important games having SNE; e.g., some congestion games (Rosenthal, 1973), as pointed out in (Holzman and Law-Yone, 1997), or economies with multilateral environmental externalities; e.g., (Nessah and Tian, 2014). Existence of such a "strong" equilibrium may result in great stability of the system, as virtually no group of players will intend to change the *status quo*. Verifying the existence (or non-existence) of such a point and locating it may be very important. Consequently, the interest in computing SNE grows, also – see; e.g., (Gatti *et al.*, 2013; Nessah and Tian, 2014).

In this paper, we consider continuous single-stage games; i.e., the case, when the player's strategy is a tuple of numbers (vector) they choose from the given set, i.e. $x^i = \left(x^i_1, \ldots, x^i_{k_i}\right) \in X_i \subseteq \mathbb{R}^{k_i}$. Let us denote K_i – the set of components of the i-th player decision variable x^i, k_i – its size, K_I – the union of all K_i for $i \in I$ and $x = (x^1, \ldots, x^n) = (x^1_1, \ldots, x^1_{k_1}, x^2_1, \ldots, x^2_{k_2}, \ldots x^n_1, \ldots, x^n_{k_n})$. Also, we call Nash points (equilibria) that are not strong, "plain" Nash equilibria, to distinguish them from SNE.

Computing Nash equilibria – plain or strong ones – of such games is a hard task in general. We are going to present an approach based on interval analysis, extending our earlier algorithm for plain Nash points; see (Kubica and Woźniak, 2010, 2012).

2. BASICS OF INTERVAL COMPUTATIONS

Now, we shall define some basic notions of intervals and their arithmetic. The idea can be found in several textbooks; e.g., (Hansen and Walster, 2004; Jaulin *et al.*, 2001; Kearfott, 1996; Moore *et al.*, 2009; Shary, 2013).

We define the (closed) interval $[\underline{x}, \overline{x}]$ as a set $\{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}$. Following (Kearfott *et al.*, 2010), we use boldface lowercase letters to denote interval variables; e.g., $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbb{IR}$ denotes the set of all real intervals.

We design arithmetic operations on intervals so that the following condition was fulfilled: if we have $\odot \in \{+, -, \cdot, /\}$, $a \in \mathbf{a}$, $b \in \boldsymbol{b}$, then $a \odot b \in \mathbf{a} \odot \boldsymbol{b}$. The actual formulae for arithmetic operations – see; e.g., (Hansen and Walster, 2004; Jaulin *et al.*, 2001; Kearfott, 1996) – are as follows:

$$
\begin{aligned}
[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \\
[\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}] \\
[\underline{a}, \overline{a}] \cdot [\underline{b}, \overline{b}] &= [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})] \\
[\underline{a}, \overline{a}] \,/\, [\underline{b}, \overline{b}] &= [\underline{a}, \overline{a}] \cdot \left[1 / \overline{b}, 1 / \underline{b}\right], \qquad 0 \notin [\underline{b}, \overline{b}]
\end{aligned}
$$

The definition of interval vector $\mathbf{x}$, a subset of $\mathbb{R}^n$ is straightforward: $\mathbb{R}^n \supset \mathbf{x} = \mathbf{x}_1 \times \cdots \times \mathbf{x}_n$. Traditionally, interval vectors are called *boxes*.

Links between real and interval functions are set by the notion of an *inclusion function*: see; e.g., (Jaulin *et al.*, 2001); also called an *interval extension*; e.g., (Kearfott, 1996).

Definition 2.1. *A function* $\mathsf{f}\colon \mathbb{IR} \to \mathbb{IR}$ *is an* inclusion function *of* $f\colon \mathbb{R} \to \mathbb{R}$*, if for each interval* $\mathbf{x}$ *within the domain of* f *the following condition is satisfied:*

$$\{f(x) \mid x \in \mathbf{x}\} \subseteq \mathsf{f}(\mathbf{x})$$

The definition is analogous for functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$*.*

When computing interval operations – either the ones above or computing the enclosure for a transcendental function – we can round the lower bound downward and the upper bound upward. This will result in an interval that will be overestimated, but will be *guaranteed to contain the true result of the real-number operation.*

3. NECESSARY CONDITIONS FOR A SNE

In interval global optimization, we use the Fritz–John conditions (Kearfott, 1996) to discard boxes that do not contain critical points. For unconstrained problems, we can discard all boxes, where the gradient of the objective cannot be equal to zero (unless *bound constraints* are active – see (Kearfott, 1996); i.e., when x belongs to the boundary of X). We are going to denote part of the gradient of a function f with respect to some of the variables $x_j = (x_1^j, \ldots, x_{k_j}^j)$ as:

$$\frac{\partial f}{\partial x^j} = \Big(\frac{\partial f}{\partial x_1^j}, \ldots, \frac{\partial f}{\partial x_{k_j}^j}\Big)$$

The vector equality $(y_1, \ldots, y_k) = 0$ is understood componentwise.

In (Kubica and Woźniak, 2010), we considered the necessary conditions of a Nash equilibrium and realized that – if no constraints are active – the point x has to satisfy the following conditions to be a Nash equilibrium:

$$\frac{\partial q_i(x)}{\partial x^i} = 0, i = 1, \ldots, n$$

What other conditions should a point satisfy to be a SNE?

Proposition 3.1. *(Necessary conditions for a 2-SNE) Consider a strategy profile x, such that no constraints are active for x (i.e., $x \in \operatorname{int} X$). Suppose, for two players i and j, we have $\frac{\partial q_i(x)}{\partial x^j} \neq 0$ and $\frac{\partial q_j(x)}{\partial x^i} \neq 0$. Then, x is not a 2-SNE.*

Interpretation

In a 2-SNE point, for no pair of players, it is possible for them to mutually improve each other's cost value – at least for one of them, their cost is minimized for the other's decision for x.

Proof. Suppose x is a 2-SNE of the game. From the definition, for each pair of players (i, j) the pair of their cost functions $\big(q_i(x), q_j(x)\big)$ has to be weakly non-dominated – see; e.g., (Nessah and Tian, 2014):

$$\nexists x' \in X \quad \big(q_i(x') < q_i(x) \text{ and } q_j(x') < q_j(x)\big)$$

Necessary conditions for the weak Pareto-optimality can be formulated as follows – see; e.g., (Miettinen, 1999) – there exist $u_1 \in [0, 1]$ and $u_2 \in [0, 1]$ such tha:

$$u_1 \cdot \frac{\partial q_i}{\partial x^i} + u_2 \cdot \frac{\partial q_j}{\partial x^i} = 0$$

$$u_1 \cdot \frac{\partial q_i}{\partial x^j} + u_2 \cdot \frac{\partial q_j}{\partial x^j} = 0$$

$$u_1 + u_2 = 1$$

From the necessary conditions of any Nash equilibrium – see; e.g., (Kubica and Woźniak, 2010) – we know that $\frac{\partial q_i}{\partial x^i} = \frac{\partial q_j}{\partial x^j} = 0$. Thus, we obtain:

$$u_2 \cdot \frac{\partial q_j}{\partial x^i} = 0$$

$$u_1 \cdot \frac{\partial q_i}{\partial x^j} = 0$$

$$u_1 + u_2 = 1$$

As $u_1 + u_2 = 1$, we cannot have $u_1 = u_2 = 0$. So, at most, one of the partial derivatives – $\frac{\partial q_j}{\partial x^i}$ or $\frac{\partial q_i}{\partial x^j}$ – has to be equal to zero. ■

Proposition 3.2. *(Necessary conditions for a 3-SNE) Consider a strategy profile x, such that no constraints are active for x (i.e., $x \in \operatorname{int} X$). Consider three players: i, j and k. A necessary condition for x to be a 3-SNE is that the following two conditions are satisfied:*

$$\begin{aligned} &\frac{\partial q_i(x)}{\partial x^j} = 0 \ \textit{or} \ \frac{\partial q_j(x)}{\partial x^k} = 0 \ \textit{or} \ \frac{\partial q_k(x)}{\partial x^i} = 0 \\ &\frac{\partial q_i(x)}{\partial x^k} = 0 \ \textit{or} \ \frac{\partial q_k(x)}{\partial x^j} = 0 \ \textit{or} \ \frac{\partial q_j(x)}{\partial x^i} = 0 \end{aligned} \tag{4}$$

Interpretation

For no trio of players, it is possible for them to mutually improve each other's cost value.

Proof. Suppose x is a 3-SNE of the game. Analogously to the previous proof, for each trio of players (i, j, k) the pair of their cost functions $\big(q_i(x), q_j(x), q_k(x)\big)$ has to be weakly non-dominated. Necessary conditions for weak Pareto-optimality can be formulated as follows in this case:

$$\begin{aligned} &u_1 \cdot \frac{\partial q_i}{\partial x^i} + u_2 \cdot \frac{\partial q_j}{\partial x^i} + u_3 \cdot \frac{\partial q_k}{\partial x^i} = 0 \\ &u_1 \cdot \frac{\partial q_i}{\partial x^j} + u_2 \cdot \frac{\partial q_j}{\partial x^j} + u_3 \cdot \frac{\partial q_k}{\partial x^j} = 0 \\ &u_1 \cdot \frac{\partial q_i}{\partial x^k} + u_2 \cdot \frac{\partial q_j}{\partial x^k} + u_3 \cdot \frac{\partial q_k}{\partial x^k} = 0 \\ &u_1 + u_2 + u_3 = 1 \end{aligned}$$

As earlier, we have $\frac{\partial q_i}{\partial x^i} = \frac{\partial q_j}{\partial x^j} = \frac{\partial q_k}{\partial x^k} = 0$, which reduces the above equations to:

$$\begin{aligned} &u_2 \cdot \frac{\partial q_j}{\partial x^i} + u_3 \cdot \frac{\partial q_k}{\partial x^i} = 0 \\ &u_1 \cdot \frac{\partial q_i}{\partial x^j} + u_3 \cdot \frac{\partial q_k}{\partial x^j} = 0 \\ &u_1 \cdot \frac{\partial q_i}{\partial x^k} + u_2 \cdot \frac{\partial q_j}{\partial x^k} = 0 \\ &u_1 + u_2 + u_3 = 1 \end{aligned} \tag{5}$$

while – from the 2-SNE's necessary conditions for each pair of players, we have:

$$\begin{aligned} &\frac{\partial q_i(x)}{\partial x^j} = 0 \ \text{or} \ \frac{\partial q_j(x)}{\partial x^i} = 0 \\ &\frac{\partial q_i(x)}{\partial x^k} = 0 \ \text{or} \ \frac{\partial q_k(x)}{\partial x^i} = 0 \\ &\frac{\partial q_j(x)}{\partial x^k} = 0 \ \text{or} \ \frac{\partial q_k(x)}{\partial x^j} = 0 \end{aligned} \tag{6}$$

Conditions (6) themselves do not assure (4) – we can choose all three partial derivatives to be equal to zero from the same line of (4). But together with (5), we can imply the following:

- We can either choose all three pairs from various equations of (5) or two of them from the same equation.
- If two pairs in the same equations of (5) are equal to zero, they are from both equations of (4).
- If all three partial derivatives are chosen from separate equations of (5), the system (5) transforms into the system of the following three equations: $u_1 \cdot a = 0$, $u_2 \cdot b = 0$, $u_3 \cdot c = 0$, where a, b and c are *different* partial derivatives. As, at most one u_i can be equal to zero, it makes at least two of the derivatives a, b and c to be equal to zero.

In all cases, derivatives from both lines of (4) are equal to zero. ■

4. THE PROPOSED APPROACH

The general schema is going to be a specific variant of the branch-and-bound type (b&b-type) method described by the author in (Kubica, 2012, 2015). The algorithm is going to seek points satisfying the logical conditions defined by (3).

The input of the algorithms is the game; i.e., the number of players, formulae for cost functions of each of them, and domains of their control variables. The program results in two sets of boxes containing "verified" and "possible" strong Nash equilibria of the game.

To process boxes in the b&b-type algorithm, we have to use the necessary conditions investigated in Section 3. Please note that these conditions form an overdetermined system. There are methods to solve overdetermined systems – e.g., (Horacek and Hladik, 2013, 2014) – but in our case, another approach seems more appropriate. We have a system of N equations in N variables (necessary conditions for a Nash point) plus additional conditions that are *alternatives of equations*.

It seems reasonable to consider the first system separately. We use the following tools to solve it:

- a variant of the monotonicity test – Algorithm 2; see also (Kubica and Woźniak, 2010),
- a variant of the "concavity" test – Algorithm 4; e.g., (Kearfott, 1996),
- an interval Newton operator (see below).

Hence, the 2-SNE necessary conditions investigated in Proposition 3.1 are used in Algorithm 3. Conditions for k-SNE, $k \geq 3$ are not checked in the current implementation – it seems to be a costly procedure and unlikely to be very useful.

Conditions from the definition of SNE – equation (3) – are directly verified in the second phase of Algorithm 1.

The "concavity" test – Algorithm 4 – could more precisely be called the "non-convexity" test. It verifies whether the function can be convex on the box $\mathbf{x}$; i.e., if no component of the Hesse matrix is negative. If bound constraints can be active, the check is not performed, as even a function that is concave with respect to some of its variables can still have a minimum on the boundaries.

The general b&b-type algorithm is implemented by Algorithm 1.

Algorithm 1 The branch-and-bound-type method for seeking SNE

Require: $x^0, \mathsf{q}(\cdot), \varepsilon$

$L_{ver} = L_{pos} = L_{check} = L_{small} = \emptyset$
$\mathbf{x} = x^{(0)}$
loop
    $\mathbf{x}^{old} = \mathbf{x}$
    perform the monotonicity test (Algorithm 2) on ($\mathbf{x}$, $\mathbf{x}^{(0)}$, $\mathbf{x}^{old}$, q)
    perform the 2-SNE-monotonicity test (Algorithm 3) on ($\mathbf{x}$, $\mathbf{x}^{(0)}$, $\mathbf{x}^{old}$, q)
    perform the "concavity" test (Algorithm 4) on ($\mathbf{x}$, $\mathbf{x}^{(0)}$, $\mathbf{x}^{old}$, q)
    perform the Newton operator on ($\mathbf{x}$, $\mathbf{x}^{(0)}$, q)
    if ($\mathbf{x}$ was discarded, but not all q_i's are monotonous on it) **then**
        push (L_{check}, $\mathbf{x}^{old}$)
        discard $\mathbf{x}$
    else if (the tests resulted in two subboxes of $\mathbf{x}$: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$) **then**
        $\mathbf{x} = \mathbf{x}^{(1)}$
        push (L, $\mathbf{x}^{(2)}$)
        cycle loop
    else if (wid ($\mathbf{x}$) $< \varepsilon$) **then**
        push (L_{small}, $\mathbf{x}$)
    end if
    if ($\mathbf{x}$ was discarded **or** $\mathbf{x}$ was stored) **then**
        $\mathbf{x} =$ pop (L)
        if (L was empty) **then**
            break
        end if
    else
        bisect ($\mathbf{x}$), obtaining $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$
        $\mathbf{x} = \mathbf{x}^{(1)}$
        push (L, $\mathbf{x}^{(2)}$)
    end if
end loop
{Second phase – verification}
for all ($\mathbf{x} \in L_{small}$) **do**
    check if another solution from L_{small} does not invalidate $\mathbf{x}$ (see Subsection 4.1)
    verify if no box from L_{check} contains a point that would invalidate $\mathbf{x}$
    put $\mathbf{x}$ to L_{ver}, L_{pos} or discard it, according to the results
end for
return L_{ver}, L_{pos}

Algorithm 2 The monotonicity test

Require: $\mathbf{x}, \mathbf{x}^{(0)}, \mathbf{x}^{old}, \mathsf{q}(\cdot)$
$n_{mon} = 0$
for $(i = 1, \ldots, n)$ **do**
  for $(k = 1, \ldots, k_i)$ **do**
    if $(\frac{\partial \mathsf{q}_i(\mathbf{x})}{\partial x_k^i} > 0)$ **then**
      if $(\underline{x}_k^i > \underline{x}_k^{i(0)})$ **then**
        increment n_{mon}
        break {the inner loop}
      else
        set $\overline{x}_k^i = \underline{x}_k^{i(0)}$
      end if
    else if $(\frac{\partial \mathsf{q}_i(\mathbf{x})}{\partial x_k^i} < 0)$ **then**
      if $(\overline{x}_k^i < \overline{x}_k^{i(0)})$ **then**
        increment n_{mon}
        break {the inner loop}
      else
        set $\underline{x}_k^i = \overline{x}_k^{i(0)}$
      end if
    end if
  end for
end for
if $(n_{mon} > 0)$ **then**
  if $(n_{mon} < n)$ **then**
    push $(L_{check}, \mathbf{x}^{old})$
  end if
  discard $\mathbf{x}$
end if

Algorithm 3 The 2-SNE monotonicity test

Require: $\mathbf{x}, \mathbf{x}^{(0)}, \mathbf{x}^{old}, \mathsf{q}(\cdot)$
for $(i = 1, \ldots, n)$ **do**
  for $(j = i + 1, \ldots, n)$ **do**
    ith_has_no_zero = jth_has_no_zero = **false**
    for $(k = 1, \ldots, k_j)$ **do**
      if $(\overline{x}_k^j < \overline{x}_k^{j(0)}$ **and** $\frac{\partial \mathsf{q}_i(\mathbf{x})}{\partial x_k^j} < 0)$ **or** $(\underline{x}_k^j > \underline{x}_k^{j(0)}$ **and** $\frac{\partial \mathsf{q}_i(\mathbf{x})}{\partial x_k^j} > 0)$ **then**
        ith_has_no_zero = **true**
      end if
    end for
    for $(k = 1, \ldots, k_i)$ **do**
      if $(\overline{x}_k^i < \overline{x}_k^{i(0)}$ **and** $\frac{\partial \mathsf{q}_j(\mathbf{x})}{\partial x_k^i} < 0)$ **or** $(\underline{x}_k^i > \underline{x}_k^{i(0)}$ **and** $\frac{\partial \mathsf{q}_j(\mathbf{x})}{\partial x_k^i} > 0)$ **then**
        jth_has_no_zero = **true**
      end if
    end for
    if (ith_has_no_zero **and** jth_has_no_zero) **then**
      discard $\mathbf{x}$
      return
    end if
  end for
end for

Algorithm 4 The "concavity" test

Require: $\mathbf{x}, \mathbf{x}^{(0)}, \mathbf{x}^{old}, \mathsf{q}(\cdot)$

$n_{conc} = 0$
if (**not** $\mathbf{x} \subset \operatorname{int} \mathbf{x}^{(0)}$) **then**
  return
end if
for $(i = 1, \dots, n)$ **do**
  {check the Hesse matrix of $\mathsf{q}_i(\mathbf{x})$ with respect to x^i}
  if $(\frac{\partial^2 \mathsf{q}_i(\mathbf{x})}{\partial (x_k^i)^2} < 0$ for some $k = 1, \dots, k_i)$ **then**
    increment n_{conc}
  end if
end for
if $(n_{conc} > 0)$ **then**
  if $(n_{conc} < n)$ **then**
    push $(L_{check}, \mathbf{x}^{old})$
  end if
  discard $\mathbf{x}$
end if

As the Newton operator, we use the interval Gauss-Seidel operator with the inverse-midpoint preconditioner. We shall not present the code, as it is available in several textbooks; e.g., (Hansen and Walster, 2004; Kearfott, 1996; Moore *et al.*, 2009; Shary, 2013).

4.1. THE SECOND PHASE – VERIFICATION

Verification of the solutions obtained in the b&b-type algorithm is based on the following property:

Property 4.1. The point $x^* = (x^{1*}, \dots, x^{n*})$ is a SNE, if $\forall x = (x^1, \dots, x^n) \in X$:

$$\begin{aligned}&\Big((\exists i = 1, \dots, n) \quad \big(q_i(x) \geq q_i(x^*)\big) \text{ and } (x^i \neq x^{i*})\Big) \text{ or} \\ &\Big((\forall i = 1, \dots, n)\, (x^i = x^{i*})\Big)\end{aligned} \tag{7}$$

Proof. From (3), we infer that no coalition $I \subseteq \{1, \dots, n\}$ can improve the objectives of all of its members. If x is the strategy profile when players cooperating in coalition I deviated from x^*, it means that, at least for one $i \in I$, the value of q_i has not improved, and for $i \notin I$, the players did not change their strategy; i.e., $x^i = x^{i*}$. ■

Proposition 4.1 means that, to invalidate x^* as a SNE, we have to find the strategy profile x such that:

$$\begin{aligned}&\Big((\forall i = 1, \dots, n) \quad \big(q_i(x) < q_i(x^*)\big) \text{ or } (x^i = x^{i*})\Big) \text{ and} \\ &\Big((\exists i = 1, \dots, n)\, (x^i \neq x^{i*})\Big)\end{aligned} \tag{8}$$

This condition can easily be checked for all other points in the list L_{small}. Boxes in L_{check} are larger, and – in general – we need to bisect them, performing a "nested" b&b--type procedure to verify if they contain a point invalidating a specific solution or not.

4.2. PARALLELIZATION

The algorithm is parallelized using threads.

In the first phase, we have a shared queue L – guarded by two mutexes, as described in (Kubica and Woźniak, 2010) – and several threads processing boxes in parallel.

In the second phase, we verify different boxes from L_{small} in parallel; i.e., we parallelize the loop in line 31. The verification procedure in line 33 (that is a nested b&b-type algorithm) is not parallelized, as – being a recursive procedure – it would require more sophisticated parallelization methods; e.g., using Intel Threading Building Blocks; see (Kubica, 2012, 2015).

5. EXAMPLES OF GAMES TO SOLVE

We are going to present results from a few test problems. The first three have been discussed in (Ślepowrońska, 1996) and then considered in (Jauernig *et al.*, 2006; Kołodziej *et al.*, 2006; Kubica and Woźniak, 2010).

The first game has two players; each of them controls one real-valued decision variable.

$$\min_{x_1} \left(q_1(x_1,x_2) = (x_1 - x_2 + 1)^2\right) \tag{9}$$
$$\min_{x_2} \left(q_2(x_1,x_2) = (x_2 - x_1^2)^2 + (x_1 - 1)^2\right)$$
$$x_1 \in [-1, 2.5],\ x_2 \in [-1, 3]$$

This game has three Nash equilibria: $(2, 3)$ on the boundary and two in the interior of the feasible set: $(-0.618034, 0.381966)$ and $(1.618033, 2.618033)$. It is not known *a priori* if they are strong or not; our solver indicates that they are. Accuracy $\varepsilon = 10^{-7}$ is used for this game.

The second game is also a game of two players, but now each of them has 9 decision variables.

$$\min_{x_1,\ldots,x_9} \big(q_1(x) = (x_1-1)^2 + (x_2-1)^2 + x_3^2 + (x_4-1)^2 + x_5^2 + (x_6-1)^2 \tag{10}$$
$$+(x_7-1)^2 + x_8^2 + x_9^2 + x_{11}^2 + (x_{12}-0.5)^2 + x_{13}^2 + (x_{16}+0.5)^2 + (x_{18}-1)^2\big)$$

$$\min_{x_{10},\ldots,x_{18}} \big(q_2(x) = (x_{10}+1)^2 + x_{11}^2 + (x_{12}-1)^2 + x_{13}^2 + x_{14}^2 + (x_{15}+1)^2$$
$$+(x_{17}-1)^2 + x_{16}^2 + (x_{18}-1)^2 + (x_2-0.5)^2 + x_3^2 + (x_4-0.5)^2 + (x_8-0.5)^2\big)$$

$$x_i \in [-2, 2.4]^9 \qquad i = 1, 2$$

The game has one Nash equilibrium: $(1, 1, 0, 1, 0, 1, 1, 0, 0, -1, 0, 1, 0, 0, -1, 0, 1, 1)$. According to our results, this point seems to be a SNE. Accuracy $\varepsilon = 10^{-4}$ is used for our solver.

In the third game, we have three players with two decision variables each.

$$\begin{aligned}
&\min_{x_1,x_2} \big(q_1(x) = (x_1+1)^2(x_1-1)^2 + (x_2+1)^2(x_2-1)^2 + x_3x_4 + x_5x_6\big) \qquad (11)\\
&\min_{x_3,x_4} \big(q_2(x) = (x_4-0.5)^2(x_4+1)^2 + (x_3+1)^2 + x_1x_2 + x_5x_6\big)\\
&\min_{x_5,x_6} \big(q_3(x) = (x_5+0.5)^2(x_5-1)^2 + (x_6-1)^2 + x_1x_2 + x_3x_4\big)\\
&x_i \in [-2, 2.4]^6 \qquad i = 1, 2, 3
\end{aligned}$$

This game has 16 Nash equilibria (they are listed in (Kołodziej *et al.*, 2006; Ślepowrońska, 1996); none of them is a SNE. We use accuracy parameter $\varepsilon = 10^{-7}$.

The fourth test problem is a game of two players; both have a single real-valued decision variable:

$$\begin{aligned}
&\min_{x_1} \big(q_1(x_1,x_2) = x_1^2 \cdot (x_1^2 - 3.75 \cdot x_1 + 3.25) + 1 + x_2^2\big) \qquad (12)\\
&\min_{x_2} \big(q_2(x_1,x_2) = x_2^2 \cdot (x_2^2 - 3.75 \cdot x_2 + 3.25) + 1 + x_1^2\big)\\
&x_i \in [-3, 3.2], \quad i = 1, 2
\end{aligned}$$

The game has a single Nash point at $(2, 2)$, but it is not a SNE – mutually deviating from 2 to 0 is beneficial for both players (but the point $(0, 0)$ is not a Nash point, at all!). Accuracy is set to $\varepsilon = 10^{-7}$.

5.1. THE GAME OF MISANTHROPIC INDIVIDUALS

This game has been proposed by the first author. Inspirations for it were congestion games (Rosenthal, 1973) and the game of dog and rabbit by Hugo Steinhaus (Steinhaus, 1960).

Consider n players, choosing their positions on a compact board – a two-dimensional domain for which we choose rectangle $D = [-3, 3] \times [-2, 2]$. Their objective is to be as far from the others as possible. Specifically, we assume that each of the players (let us give him the number $i = 1, \ldots, n$) maximizes, by choosing position $(x_i, y_i) \in D$, the following function:

$$q_i(x_i, y_i) = \sum_{j=1, j\neq i}^{n} \big((x_i - x_j)^2 + (y_i - y_j)^2\big) \qquad (13)$$

Solutions of the game

Depending on n, the game can have different numbers of Nash equilibria – all or none of them being strong.

For two players, we have 4 Nash equilibrium points, each of them are strong. Their structures are obvious: one of the individuals is located in one of the four corners and the other one – diagonally opposite to him. It is clear that all of them are SNE – cooperation of both players cannot increase their distance in any way. This case is a "degenerate" case of a game, as both players maximize the same function – the (square of) the distance between them.

For three players we have 36 Nash equilibria: 24 with all three individuals located in different corners ($4 \times 3 \times 2$) and 12 with one of the three individuals in a corner and both others diagonally opposite to him (one of the 3 individuals $\times$ 4 corners). In all cases, one of the individuals has a better position than the two others. And actually, none of these solutions is strong – the two players with worse values can always collude to change their positions and improve their payoffs at the expense of the third player.

For four players, we have 36 Nash equilibria: 24 solutions with each individual in his own corner ($4 \times 3 \times 2$) and 12 solutions with two pairs of players in opposite corners. Counter-intuitively, formula (13) makes their values identical for both types of solutions. All of these 36 solutions are strong Nash equilibria.

For larger number of players, it is very difficult to analyze all possible solutions and their structures. In Section 6, we present; i.a., computational results for such situations (Tables 2 and 3).

6. NUMERICAL EXPERIMENTS

Numerical experiments were performed on a computer with four cores (allowing hyper-threading), namely, an Intel Core i7-3632QM with 2.2GHz clock. The machine ran under control of a 64-bit Manjaro 0.8.8 GNU/Linux operating system with the GCC 4.8.2, glibc 2.18 and the Linux kernel 3.10.22-1-MANJARO.

The solver is written in C++ and compiled using the GCC compiler. The C-XSC library (version 2.5.3) (C-XSC, 2013) was used for interval computations.

The parallelization was done using the threads of the C++11 standard. OpenBLAS 0.2.8 (OpenBLAS, 2013) was linked for BLAS operations.

6.1. RESULTS FOR PROBLEMS (9)–(11)

Computational results for these problems can be found in Table 1.

Table 1. *Computational results for the solver, with a single thread*

problem	(9)	(10)	(11)	(12)
cost fun. evals	26557	356776	0	238
gradient evals	6875	93914	0	134
Hesse matrix evals	204	182	609	162
bisections	49	45	101	29
deleted monot. test.	35	45	67	21
deleted strong mon.	0	0	35	1
deleted “conc.”	0	0	0	3
deleted Newton	0	0	0	7
boxes after 1st ph.	3	1	0	3
possibly dominating	41	45	102	32
deleted 2nd phase	0	0	0	3
possible solutions	3	1	0	0
verified solutions	0	0	0	0
time (milisec.)	491	2358	459	461

6.2. RESULTS FOR THE GAME OF MISANTHROPIC INDIVIDUALS

We present results for computing SNE and plain Nash equilibria – Tables 2 and 3, respectively. Accuracy $\varepsilon = 10^{-8}$ is set in all cases.

Table 2. *Computational results for the solver, with four threads*

players number	2	3	4	5	6	7
cost fun. evals	7616	1164	4853056	70235	5576803158	1519735
gradient evals	0	0	728800	0	1210016856	0
Hesse matrix evals	3774	18141	71164	300555	1136634	4677113
bisections	943	3023	8895	30055	94719	334079
deleted monot. test.	0	0	220	168	256	1536
deleted strong mon.	0	0	0	0	0	0
deleted "conc."	928	2960	8640	28864	90368	316160
deleted Newton	0	0	0	0	0	0
boxes after 1^{st} ph.	16	64	256	1024	4096	16384
possibly dominating	944	3280	10304	41972	133120	516864
deleted 2^{nd} phase	12	64	220	1024	3696	16384
possible solutions	0	0	36	0	400	0
verified solutions	4	0	0	0	0	0
time (sec.)	0.452	0.555	4.442	5.577	5221	189

Table 3. *Computational results for computing plain Nash equilibria, using four threads*

players number	2	3	4	5	6	7
cost fun. evals	5196	47335	47602	685225	1111178	14443406
gradient evals	0	0	0	0	0	0
Hesse matrix evals	3774	18141	71164	300555	1136634	4677113
bisections	943	3023	8895	30055	94719	334079
deleted monot. test.	0	0	0	168	256	1536
deleted "conc."	928	2960	8640	28864	90368	316160
deleted Newton	0	0	0	0	0	0
boxes after 1^{st} ph.	16	64	256	1024	4096	16384
possibly dominating	944	3280	10304	41972	133120	516864
deleted 2^{nd} phase	12	28	220	624	3696	11484
possible solutions	0	32	0	336	0	4000
verified solutions	4	4	36	64	400	900
time (sec.)	0.474	0.581	1.220	7.296	37	483

7. ANALYSIS OF THE RESULTS

The algorithm finds the SNE in all cases, but it is rarely able to verify them. The conditions are a bit complicated to be verified rigorously – actually, the verification was successful in one case only – and a very specific one (the game of misanthropic individuals, $n = 2$).

The solver finds the solution for problem (11) very quickly. This is because, for this game, all 16 Nash equilibrium points can be verified *not* to be SNE early – in the first phase (see the row "boxes after 1st ph."), using the *2-SNE monotonicity test* (Algorithm 3). See Table 1 for specific results.

In the game of misanthropic individuals – problem (13) – for some n's, the number of gradient evaluations is equal to zero. This usually happens when the number of points to verify in the second phase is equal to zero.

Gradients are computed in two cases:

- in the interval Newton operator, verifying first-order conditions for the Nash equilibria,
- in the second phase – also, in the Newton operator, but now verifying the inequality that $q_i(\mathbf{x})$ is lower than the verified value.

For the game of misanthropic individuals, cost functions q_i are concave, so we never apply the Newton operator in the first phase. Nor we do in the second phase, if there are no solutions to verify (in the case, we simply do nothing in the second phase).

It is worth noting how the computational effort changes with n for the game of misanthropic individuals. For odd numbers of players ($n = 3, 5, 7$), the effort of finding all strong Nash equilibria is particularly low. The reason is simple – there is no SNE (this hypothesis has been verified by numerical experiments for $n = 3, 5, 7$; we haven't proven it for other values of n, but it seems plausible) and all possible solutions are quickly discarded by comparisons with other possible solutions (see Subsection 4.1). The time-consuming nested branch-and-bound type procedure does not have to be executed at all. Because of this, the solver for SNE is more efficient than for plain Nash equilibria for these points; see Tables 2 and 3.

For $n = 6$, the solver finds 400 points that are strong Nash equilibria, probably. Isolating so many solutions of the game is possible thanks to the virtues of the interval calculus: see; e.g., (Kubica, 2015; Shary, 2013).

8. CONCLUSIONS

We presented an interval solver able to compute strong Nash equilibrium points of continuous games. We tested it on a few test problems, showing its usefulness. Parallelization using threads allows us to handle relatively difficult problems. For one of the examples, it allowed us to isolate 400 equilibrium points.

Also, a specific test problem has been proposed – the game of misanthropic individuals; a continuous game with an arbitrary number of players, having various numbers of plain and strong Nash equilibria, depending on the number of players. It seems an interesting benchmark, due to its complex and counter-intuitive properties.

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Game-Theoretic Approach to Bank Loan Repayment

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Abstract. This paper presents a model of bank-loan repayment as a signaling game with a set of discrete types of borrowers. The type of borrower is the return on an investment project. A possibility of renegotiation of the loan agreement leads to an equilibrium in which the borrower adjusts the repaid amount to the liquidation value of its assets (from the bank's point of view). In the equilibrium, there are numerous pooling equilibrium points, with values rising according to the expected liquidation value of the loan. The article additionally proposes a mechanism forcing the borrower to pay all of his return instead of the common liquidation value of subset of types of the borrower. The paper contains also a simple numerical example explaining this mechanism.

Keywords: bank, loan, credit agreement, repayment, renegotiation, game theory

Mathematics Subject Classification: 91A28, 91A80, 91B44

JEL Classification: C72, D86, G21

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1. INTRODUCTION

A bank, granting a loan, has no possibility of influencing its actual spending; hence, a precise credit agreement should be signed. The main elements of a loan agreement are: the amount of the loan, the amount of repayment and collateral. Due to the asymmetry of information, a necessary component of the loan agreement is collateral, imposing on the debtor an incentive to repay the loan. However, the option to renegotiate the loan agreement, changes the relationship between the bank and the borrower.

The possibility of renegotiation inclines the borrower to transfer a part of the risk to the lender. The research belonging to the theory of incomplete contracts suggest that the borrower renegotiate a contract or even does not make repayment if the liquidation value of collateral is low. This idea is incorporated, among others, in the works of: Aghion and Bolton (1992), Bester (1994), Hart and Moor (1994 and 1998), Bolton and Scharfstein (1996), Lacker (2001) and Paliński (2013).

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According to this research, the liquidation value of collateral determines not only the amounts recovered by the bank in the absence of repayment of the loan, but also affects the results of renegotiation of the debt. This is due to the fact that the threat of liquidation of assets motivates the borrower to avoid insolvency; consequently, the liquidation value determines the amount of debt repayment. When the liquidation value is low – the bargaining power of the borrower increases and the amount of debt repayment reduces.

The main aim of this article is to analyze a model of loan repayment as a signaling game in which the borrower sends a signal – a proposed amount of repayment – whereas the bank receives the signal and takes action – accepts or rejects the proposal (Cho and Kreps, 1987; Fudenberg and Tirole, 1991). Depending on the liquidation value of the borrower's assets, the possibility of the debt renegotiation plays a crucial role in the model. The presented model refers to the issue of the role of collateral for the repayment of the loan and the renegotiation considered by Bester (1994). In addition, a voluntary repayment of the loan is assumed in the model, in which the borrower determines the amount of repayment, trying to avoid enforcement of the debt. This approach refers to models of Krasa–Villamil (2000) and Krasa–Sharma–Villamil (2005).

The remainder of this paper is organized as follows: Section 2 presents the assumptions of the model and the analysis of the equilibrium in the model. Section 3 contains a proposition of an incentive mechanism, motivating the borrower to repay the highest possible amount. Section 4 provides main conclusions.

2. THE MODEL

Consider an economy with two risk neutral agents: an entrepreneur and a bank (their indexes are, respectively, E and B). Assume that the entrepreneur signed a credit agreement with the bank to finance a venture. The credit agreement (I, R_1, C) with the bank is a triplet that with a given amount of the loan I determines the amount of repayment R_1 and collateral C. Loan repayment R_1 is independent of realization of the project; thus, we assume that the credit agreement as a *standard debt contract* (SDC) with collateral. The standard debt contract is a contract in which a borrower agrees to pay a fixed amount, and non-payment allows a bank to seize the borrower's assets being the output of the project.

The return of the project is a random variable Y with discrete realizations $\mathrm{y} \in \mathrm{W} = \{0, \ldots, \overline{\mathrm{y}}\} \subset \mathbb{R}_{+}$, where $\overline{\mathrm{y}} > 0$, with cumulative distribution $F(y)$. This return of the project is observable without costs only to the entrepreneur, what is known to both agents before signing the agreement. Agents have common knowledge of the prior beliefs $\beta(\cdot)$ on the set of possible realizations of Y, where $\beta(y) > 0$.

In case of default, the entrepreneur can turn to the bank for debt restructuring involving a cancellation of the debt. The bank may restructure the debt or seize collateral together with the venture.

The value of collateral is lower for the bank than for the borrower because of the cost of acquisition and liquidation, and is bC, where $0 \leq b < 1$. Similarly, the acquisition and management of the project generates considerable costs, thus the value of the project for the bank is aY, where $0 \leq a < 1$. Assume further that $bC \leq R_1$, which means that the bank does not have collateral exceeding the value of the loan. Assume also a reservation utility of the bank $\overline{u}$, which can be positive for an insolvent borrower or negative for a promising borrower. Let us define the value of the borrower's asset from the bank's point of view in the case of default as follows.

Definition 2.1. *The value $L(y) = ay + bC + \overline{u}$ is called the liquidation value of loan.*

After signing the credit agreement $(I,\ R_1,\ C)$ in period $t = 0$, the game of loan repayment is as follows.

1) In the first period $t = 1$, nature selects the return of the project y.
2) In the next period $t = 2$, the entrepreneur observes the realization of the project. Knowing the return of the project the entrepreneur decides to make voluntary payment $v \in V = [0, \overline{y}] \subset \mathbb{R}_{+}$. If $v < R_1$, then the entrepreneur declares default and counts on debt forgiveness. The pure strategy of the entrepreneur is the payment v made after observing the return of the project y. The decision on the payment v can no longer be changed at a later date (due to the inclusion of relevant data in financial books).
3) In the last period $t = 3$, the bank observes proposed v, but does not know the true state of nature. When $v < R_1$, the bank decides whether to restructure the debt and sign a new credit agreement $(I,\ v,\ C)$ or to seize the project with collateral. If proposed $v = R_1$, the bank accepts payment in accordance with the credit agreement. The behavior strategy of the lender $\sigma_B \in \sum_B = [0, 1]$ is the probability of loan restructuring and acceptance of payment v. The strategy $\sigma_B = 1$ means the acceptance of the payment v; moreover, if $v < R_1$ – this means a new contract and partial cancellation of the debt on amount $x = R_1 - v$. While $\sigma_B = 0$ means the liquidation of the loan.

Expected payoff of the entrepreneur $E\pi_E$ and the bank $E\pi_B$ for the pair of strategies v and σ_B after observing the return y by the entrepreneur are as follows:

$$\mathrm{E}_{v,\sigma_\mathrm{B}} \pi_E(y, v, \sigma_B) = \sigma_B(y - v) - (1 - \sigma_B)C \tag{1}$$

and:

$$\mathrm{E}_{\sigma_\mathrm{B}} \pi_B(y, \nu, \sigma_B) = \sum_{y \in W} \beta(y|v)[\sigma_B v + (1 - \sigma_B)(ay + bC)] \tag{2}$$

An updated belief $\beta(y|v)$ is the probability assigned by the bank to the type of the entrepreneur y after observing the proposed payment v.

Definition 2.2. *A strategy profile v, σ_B along with beliefs $\beta(y)$, $\beta(y|v)$ is a perfect Bayesian equilibrium if and only if:*

(i) *$v \in V$ maximizes $E_{v,\sigma_B} \pi_E(y, v, \sigma_B)$ for every y.*
(ii) *$\sigma_B \in \sum_B$ maximizes $E_{\sigma_B} \pi_B(y, v, \sigma_B)$ for every v.*
(iii) *$\beta(y|v) = \frac{v(y)\beta(y)}{\sum_{y' \in Y}[v(y')\beta(y')]}$ if it is possible. Otherwise $\beta(y|v)$ is any probability on $\{y \in W | y \geq v\}$.*

Conditions (i) and (ii) impose a requirement that each strategy was perfect Bayesian equilibrium for each subgame with some beliefs. The return on the project y is, at the same time, the type of entrepreneur in the game. Condition (iii) specifies how to update beliefs after observing the amount of repayment υ using Bayes's rule (Bayes, Price, 1763; see: Fudenberg and Tirole, 1991).

The entrepreneur, knowing the return on the project y and deciding to make a payment υ, has to solve the following problem.

Problem 2.1. At time $t = 2$ find υ and σ_B to solve the following optimization problem

$$\max_{\upsilon,\sigma_B} E_{\upsilon,\sigma_B}\pi_E\left(y,\upsilon,\sigma_B\right) \tag{3}$$

subject to:

$$E_{\sigma_B}\pi_B\left(y,\upsilon,\sigma_B\right) \geq \sum_{y\in W} \beta(y|\upsilon)\min\left\{R_1,\ L\left(y\right)\right\} \text{ for each } y,$$

$$\text{having } \beta\left(y|\upsilon\right) > 0 \tag{4}$$

$$0 \leq \upsilon \leq y \text{ and } \upsilon \leq R_1 \text{ for all } y \tag{5}$$

$$\upsilon, \sigma_B, \beta\left(y\right), \beta\left(y|\upsilon\right) \text{ are a perfect Bayesian equilibrium at } t = 3 \tag{6}$$

The individual rationality constraint of the bank (4) indicates that in case of payment lower than the payment R_1 required by the credit agreement the bank in the worst case is assured of the expected liquidation value of the loan.

Condition (5) imposes a requirement on the voluntary payment υ that it be no greater than the return on the project y and does not exceed R_1.

Definition 2.3. *Let $\underline{y_k} \in W$ is an arbitrary chosen type of the borrower. The expected liquidation value of the loan, EL_k, for types $y \in W$, $y \geq \underline{y_k}$, is defined as follows:*

$$EL_k = \mathop{max}_{y\in W} \frac{\sum\limits_{y\geq \underline{y_k}} \beta\left(y\right)\left(ay + bC + \overline{u}\right)}{\sum\limits_{y\geq \underline{y_k}} \beta\left(y\right)} \leq \underline{y_k} \tag{7}$$

where $\underline{y_1} = \mathop{min}\limits_{y\geq \frac{bC+\overline{u}}{(1-a)}} y$, $\underline{y_{k+1}} = \mathop{min}\limits_{y>\underline{y_k}} y$ *such that* $EL_k > \underline{y_k}$, $k = 1, 2, 3, \ldots, n$.

The expected liquidation value EL_k is defined as the maximum expected value of liquidation of some subset of types greater or equal than $\underline{y_k}$, for which this value is not greater than the return on the project of the lowest type $\underline{y_k}$ in the subset. The $\underline{y_k}$ belongs to a discrete set of types, whereas the expected liquidation value EL_k is a real number; therefore, there is a weak inequality in formula (7). The next expected liquidation value EL_{k+1} can be calculated for a subset of types greater than the largest type in the previous subset.

Proposition 2.1. *The solution of Problem 2.1 is the following perfect Bayesian equilibrium in which*

1) *The strategy of the borrower is:*

$$v(y) = \begin{cases} 0, & y < \dfrac{bC + \overline{u}}{(1-a)}, \\ EL_k, & \dfrac{bC + \overline{u}}{(1-a)} \le y < \dfrac{R_1 - bC - \overline{u}}{a}, \quad k = 1, 2, 3, \dots, n, \\ R_1, & y \ge \dfrac{R_1 - bC - \overline{u}}{a}. \end{cases}$$

2) *The strategy of the bank is:*

$$\sigma_B(v) = \begin{cases} 0, & v < \dfrac{bC + \overline{u}}{(1-a)} \\ 1, & v \ge \dfrac{bC + \overline{u}}{(1-a)} \end{cases}$$

Proof. Consider a borrower whose return is less than his liquidation value of the loan $L(y)$, that is $y < ay + bC + \overline{u} = L(y)$. Such borrower is able to pay $v \le y < L(y)$. An updated belief of the borrower's type is $\beta\left(y < (bC + \overline{u})/(1-a)|v\right) = 1$, so the strategy of the bank is liquidation $\sigma_B = 0$ and the strategy of the borrower is not to pay, which satisfies conditions (i) and (ii) of Definition 2.2.

Consider a borrower $\underline{y_1} \in W$, whose return is equal to or greater than his liquidation value, such that $\underline{y_1} = \min_{\frac{bC + \overline{u}}{1-a} \le y \le \overline{y}} y$. He is able to pay $v \ge (bC + \overline{u})/(1-a)$ to avoid liquidation. Consider further types of borrower $y \in \left\{\underline{y_1}, \dots, \overline{y_1}\right\} \subset \left[\frac{bC + \overline{u}}{1-a}, \overline{y}\right]$, where $\overline{y_1} \ge \underline{y_1}$, for which the expected liquidation value is $\frac{1}{\sum_{y \in \left\{\underline{y_k}, \dots, \overline{y_k}\right\}} \beta(y)} \left[\sum_{y \in \left\{\underline{y_1}, \dots, \overline{y_1}\right\}} \beta(y)\, ay + bC + \overline{u}\right] = EL_1 \le \underline{y_1}$. Rather than pay an amount equal to their return on investment y (which will give them an income equal zero), they can pretend a lower return and pay $v = EL_1$, which gives them $y - EL_1 \ge 0$. On the basis of the proposed repayment v, the bank is not able to distinguish the type of borrower and an updated belief is $\beta\left(y \ge (bC + \overline{u})/(1-a)|v = EL_1\right) = 1$. As the expected liquidation value is $EL_1 = v$, the bank will accept this payment, and will use a strategy $\sigma_B(v) = 1$. Liquidation would give the bank no more.

None of the types of borrower can individually change the strategy to repay less than the liquidation value of the common EL_1, even if its own liquidation value is below the common value EL_1, because in the absence of correlation with other types, this would reduce the value of the repayment below the expected liquidation value. In case $EL_1 > v$, the bank would prefer liquidation and repayment v less than the liquidation value EL_1 by the borrower of any type would result in the change of bank's strategy to $\sigma_B(v) = 0$, which would give the borrower income $-C$. Hence, repayment $v < EL_1$ is not the optimal strategy of the borrower.

Consequently, both strategies satisfy conditions (i) and (ii) of Definition 2.2. There exist conditions for pooling equilibrium in which borrower of types $y \in \{\underline{y_1}, \ldots, \overline{y_1}\}$ pay the same amount $v = EL_1$.

If, however, the expected liquidation value of the loan EL_1 for all types of $y \geq (bC + \overline{u})/(1-a)$ was greater than $\underline{y_1}$, then for types $y > \overline{y_1}$ must be created another equilibrium $v = EL_2 > \underline{y_1} \geq EL_1$ providing loan restructuring. Otherwise, the bank would prefer to liquidate, giving him a higher expected payoff, rather than restructuring the debt.

This reasoning should be repeated for the next EL_k until the expected liquidation value of the loan for a certain type y exceeds repayment R_1 required in the loan agreement. Substituting y with R_1 in the formula of the liquidation value we get $y \geq (R_1 - bC - \overline{u})/a$. Above this value, the borrower repays the amount $v = R_1$, and the bank has to accept payment according to the loan agreement. ■

Figure 1 depicts a diagram of debt repayment depending on the amount of return on the project resulting from Proposition 2.1. Repayment of the loan R_1, specified in the loan contract, begins not when return is $y = R_1$, but when the expected liquidation value of the loan is equal to R_1, that is for y satisfying the inequality $ay + bC + \overline{u} \geq R_1$.

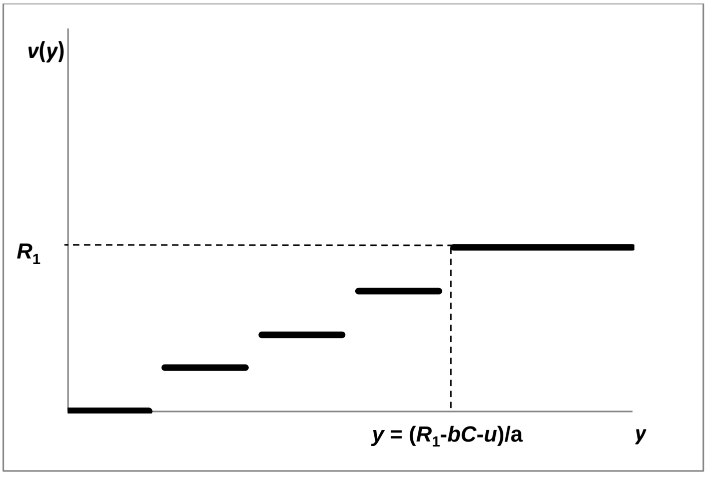

Fig. 1. *Loan repayment diagram resulting from Proposition 2.1*

3. EXTENSION OF THE MODEL – FUTURE LOANS

There exists a great amount of literature the on so-called *relationship lending*. Such a term covers the economic value of long-term cooperation between a bank and a borrower – an important role in this field plays the model of Rajan (1992) (see Boot, 2000 for overview of the models; examples of empirical studies can be found in works of: Petersen and Rajan, 1994, Degryse, 2000, Bharath *et al.*, 2007). Omitting complex interdependencies of such relationship, each entrepreneur thinking about further operating activity, needs future loans.

Let us assume that the enterprise – current borrower – has ability to run next project that can give him the expected payoff $E\pi_2$. The enterprise requires external sources of financing to run this project. Taking into account the finding of Proposition 2.1. we would like to construct a mechanism that would force the borrower to pay all his return on the first project instead of the expected liquidation value. If we made the probability of granting the next loan by the incumbent bank dependent on the amount repaid for the first loan we would achieve the mechanism of true payment.

Assumption 3.1. Let m be the probability of granting the next loan. Correct mechanism $m(v)$ depending on the amount of the repayment of the previous loan must meet the following conditions:

1) Repayment $v \le (bC + \overline{u})/(1-a)$ means taking over collateral and usually results in bankruptcy of the borrower – not giving a chance to grant another loan, i.e. $m\left(v \le (bC + \overline{u})/(1-a)\right) = 0$.
2) Repayment required in the loan agreement and equal to R_1 is reworded in certainty of getting another loan, i.e. $m(v = R_1) = 1$.
3) Repayment between these extreme values $v \in \left(\frac{bC+\overline{u}}{1-a}, R_1\right)$ allows for another loan with the probability that is a linear function of the repaid amount on the interval specified by extreme values. This is due to the fact that a standard debt contract for $y < R_1$ is a linear function of the return on the business venture and is $R(y) = y$ (cf. Gale and Helwig, 1985).

   Condition 3) should ensure that $v(y) = y$.

Lemma 3.1. *The mechanism satisfying Assumption 3.1. is described by the following function:*

$$m(v) = \begin{cases} 0, & v < \dfrac{bC+\overline{u}}{(1-a)} \\ \dfrac{v - \frac{bC+\overline{u}}{1-a}}{R_1 - \frac{bC+\overline{u}}{1-a}}, & v \ge \dfrac{bC+\overline{u}}{(1-a)} \end{cases}$$

Proof.

(1) $m\left(v = \frac{bC+\overline{u}}{1-a}\right) = 0$.

(2) $m(v = R_1) = 1$.

(3) The function $m(v)$ is linear for $v \in \left[\dfrac{bC+\overline{u}}{1-a}, R_1\right]$. ■

The effectiveness of such a mechanism to incentivize the borrower to allocate the total return on the project y for repayment of the loan will depend on the value of the expected return on the next investment project $E\pi_2$. The expected value of a new project from the point of view of the entrepreneur is also dependent on the probability of granting the next loan, without which it will be impossible to undertake the project due to a lack of funds.

Proposition 3.1. *Mechanism $m(v)$ is effective when:*

$$E\pi_2 \ge \frac{1}{1-a}\left[R_1 - L(R_1)\right].$$

Proof. According to Proposition 2.1., some types of borrower try to form pooling equilibrium υ, for which their total expected liquidation value is no less than their possible payment υ. The payment υ is lower than or equal to return y for each type forming the pooling equilibrium.

To avoid forming an equilibrium, it is necessary for each type of borrower that the expected value of payoff from the next investment project after repayment of the whole return of the current project y is not less than the excess return y over the payment υ. For this purpose two conditions must be fulfilled:

1) $m(y)\,E\pi_2 \geq y - L(y)$, for each $\frac{bC+\overline{u}}{1-a} \leq y \leq R_1$.
2) $[m(y'') - m(y')]\,E\pi_2 \geq y'' - y'$, for each $y^{''} > y^{'}$, where $y^{''}, y^{'} \in W$, $\frac{bC+\overline{u}}{1-a} \leq y' \leq R_1$, $\frac{bC+\overline{u}}{1-a} \leq y'' \leq R_1$.

The first condition is that the lowest possible payment υ equal to the expected liquidation value for any type is not profitable. The second condition implies that if $EL(y^{''}) \geq y^{'}$, the strategy of pretending lower return and payment $\upsilon = y^{'}$ is also unprofitable.

Starting from Condition (1) and substituting $m(\upsilon)$ with function from Lemma 3.1., we have:

$$m(y)\,E\pi_2 = \frac{y - \frac{bC+\overline{u}}{1-a}}{R_1 - \frac{bC+\overline{u}}{1-a}} E\pi_2 \geq y - L(y) = y - \; ay + bC + \overline{u}$$

which, after transformation, gives:

$$E\pi_2 \geq (1-a)\,R_1 - \; bC - \overline{u} = R_1 - L(R_1)$$

Considering Condition (2) we have:

$$[m(y'') - m(y')]\,E\pi_2 = \left[\frac{y'' - \frac{bC+\overline{u}}{1-a}}{R_1 - \frac{bC+\overline{u}}{1-a}} - \frac{y' - \frac{bC+\overline{u}}{1-a}}{R_1 - \frac{bC+\overline{u}}{1-a}}\right] E\pi_2 \geq y^{''} - y^{'}$$

Rearranging we get:

$$\left(y^{''} - y'\right) E\pi_2 \geq (y'' - y')\left(R_1 - \frac{bC+\overline{u}}{1-a}\right)$$

which leads to:

$$E\pi_2 \geq R_1 - \frac{bC+\overline{u}}{1-a} = \frac{1}{1-a}\left[(1-a)\,R_1 - bC - \overline{u}\right]$$

and finally gives:

$$E\pi_2 \geq \frac{1}{1-a}\left[R_1 - L(R_1)\right]$$

Since $1/(1-a) \geq 1$, Condition (2) imposes a higher or equal value on $E\pi_2$ than Condition (1); therefore, Condition (2) is binding. ■

Mechanism $m(v)$ appears to be effective when the expected payoff from a future project is not less than the difference between the repayment under a loan agreement R_1 and the repayment of the liquidation value for y equal to repayment R_1 multiplied by the reciprocal of the coefficient of the loss of value of the project. In the case of the standard debt contract return on the project $y = R_1$ is the border point, below which there is an area of insolvency.

The result of Proposition 3.1. is not very optimistic. The expected value of a new project planned by the borrower must be high so that the borrower would spend the whole return from the first project to repay the loan. This is illustrated by a simple numerical example. Let us take the following assumptions: $R_1 = 100$, $C = 100$, $\overline{u} = 0$. The minimum $E\pi_2$, value depending on the recovery coefficients a and b, is shown in Table 1.

Table 1. *The minimum expected payoff of the next project $E\pi_2$ incentivizing the borrower to repay the first loan depending on recovery coefficients a and b*

		Coefficient a								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Coefficient b	0.1	88.89	87.50	85.71	83.33	80.00	75.00	66.67	50.00	–
	0.2	77.78	75.00	71.43	66.67	60.00	50.00	33.33	–	–
	0.3	66.67	62.50	57.14	50.00	40.00	25.00	–	–	–
	0.4	55.56	50.00	42.86	33.33	20.00	–	–	–	–
	0.5	44.44	37.50	28.57	16.67	–	–	–	–	–
	0.6	33.33	25.00	14.29	–	–	–	–	–	–
	0.7	22.22	12.50	–	–	–	–	–	–	–
	0.8	11.11	–	–	–	–	–	–	–	–
	0.9	–	–	–	–	–	–	–	–	–

Under normal economic conditions, most common recovery rates are of 0.2–0.5, the profitability of assets is equal from several to more than ten percent, and the share of debt in financing a project is about half of the investment funds. In this situation, the next business project would have to be several times larger than the previous project to ensure the required expected payoff, which is hardly possible in practice.

4. CONCLUSIONS

The key role for the repayment of the bank loan plays the liquidation value of assets of the borrower. In the game of loan repayment, a perfect Bayesian equilibrium is formed from a plurality of pooling equilibria in which the amount of the repayment of the loan is equal to the common expected liquidation value of several types. Each following point of pooling equilibrium is created by a subset of types of the borrower with a higher common expected liquidation value.

We can construct a mechanism motivating the borrower to use the total return on the investment project to repay the loan, if the repayment is not in accordance with the loan agreement. This mechanism is based on the dependence of the probability of granting a future loan on the repayment of the former loan. Unfortunately, the effectiveness of this mechanism is limited by the high expected value of the future investment project.

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Hyperbolicity of Systems Describing Value Functions in Differential Games which Model Duopoly Problems

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Abstract. Based on the Bressan and Shen approach (Bressan and Shen, 2004; Shen, 2009), we present an extension of the class of non-zero sum differential games for which value functions are described by a weakly hyperbolic Hamilton–Jacobi system. The considered value functions are determined by a Pareto optimality condition for instantaneous gain functions, for which we compare two methods of the unique choice Pareto optimal strategies. We present the procedure of applying this approach for duopoly.

Keywords: duopoly models, semi-cooperative feedback strategies, Pareto optimality, hyperbolic partial differential equations

Mathematics Subject Classification: 91A23, 49N70, 49N90, 35L65

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1. INTRODUCTION

Dynamic models describe situations in which two or more players make their decisions about their own behavior under the same circumstances. In this paper, we shall consider games with a finite duration of time. We shall be interested in solving theoretical maximizing problems that can be applied to finding better strategies in models of duopoly. Our effort is focused on finding a better solution than the Nash equilibrium. On the one hand, we want the solution to provide greater payoffs for both players, but also we want to obtain a well-posed system of PDEs describing value functions.

We assume that the evolution of state is described by the following differential equation:

$$\dot{x} = f(x) + \phi(x)u_1 + \psi(x)u_2 \tag{1}$$

with initial data:

$$x(\tau) = y \in \mathbb{R}^m \tag{2}$$

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where $f:\mathbb{R}^m\to\mathbb{R}^m$, $\phi,\psi:\mathbb{R}^m\to\mathbb{M}_{m\times n}(\mathbb{R})$ and u_i are feedback strategies – they depend on time t and state $x(t)$, $(i=1,2)$. The goal of the *i-th* player is to maximize his payoff function; i.e.,

$$J_i(\tau,y,u_1,u_2)=g_i(x(T))-\int_\tau^T h_i(x(t),u_i(t))dt \tag{3}$$

where

$$\text{a terminal payoff } g_i:\mathbb{R}^m\to\mathbb{R} \text{ is a non negative and smooth function} \tag{4}$$

and

$$\begin{gathered}\text{a running cost } h_i:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R} \text{ is a smooth function such that}\\ h_i(x,\cdot) \text{ is strictly convex for every } x\in\mathbb{R}^m\end{gathered} \tag{5}$$

We consider the instantaneous gain functions:

$$\begin{aligned}Y_1(x,p_1,p_2,u_1,u_2)&=p_1\cdot F(x,u_1,u_2)-h_1(x,u_1)\\ Y_2(x,p_1,p_2,u_1,u_2)&=p_2\cdot F(x,u_1,u_2)-h_2(x,u_2)\end{aligned} \tag{6}$$

where $F(x,u_1,u_2)$ is the right side of the dynamic and the dot denotes the scalar product (in (1), we have $F(x,u_1,u_2)=f(x)+\phi(x)u_1+\psi(x)u_2$). Fixing $x,p_1,p_2\in\mathbb{R}^m$ and $s>0$, we can find Pareto optimal[1] choices $U_i^P(x,p_1,p_2,s)$ for the static game $Y_i(x,p_1,p_2,\cdot,\cdot)$, $i=1,2$, in the following way: if (u_1^P,u_2^P) is the maximum of the combined payoff $Y_s=sY_1+Y_2$, then the strategies (u_1^P,u_2^P) give Pareto optimal payoffs in game (Y_1,Y_2). As a result, strategies U_i^P depend on s. We choose a smooth function $s(x,p_1,p_2)$ and define feedback strategies $U_i^s(x,p_1,p_2)=U_i^P(x,p_1,p_2,s(x,p_1,p_2))$ for the problem (1)–(3). Such strategies are called semi-cooperative (Bressan and Shen, 2004). If functions:

$$V_i^s(\tau,y)=J_i(\tau,y,U_1^s,U_2^s)$$

are smooth enough, then they satisfy the following system:

$$\begin{cases}V_{1,t}+H_1(x,\nabla_xV_1,\nabla_xV_2)=0\\ V_{2,t}+H_2(x,\nabla_xV_1,\nabla_xV_2)=0\end{cases} \tag{7}$$

with the terminal data:

$$V_1(T,x)=g_1(x)\qquad\text{and}\qquad V_2(T,x)=g_2(x) \tag{8}$$

where the Hamiltonian functions are given by:

$$H_i(x,p_1,p_2)=Y_i(x,p_1,p_2,U_1^s(x,p_1,p_2),U_2^s(x,p_1,p_2))$$

and they depend on s. Functions V_i^s are usually called the value functions.

[1] We say that (u_1^P,u_2^P) is a pair of Pareto optimal choices for the game, which is given by payoff functions $Y_i(u_1,u_2)$ $(i=1,2)$, if there exists no pair (u_1,u_2) such that $Y_1(u_1,u_2)>Y_1(u_1^P,u_2^P)$ and $Y_2(u_1,u_2)>Y_2(u_1^P,u_2^P)$. This means that no pair of admissible strategies exists that improve both payoffs simultaneously.

In Section 2, we shall prove that system (7) is weakly hyperbolic and is hyperbolic except for some curves on the (p_1, p_2)-plane (see Theorem 2.1). If the system is hyperbolic, then it is well-posed and this fact is crucial for numerical solutions (for more details, see (Serre, 2000)). This result is a generalization of Theorem 3 from (Bressan and Shen, 2004). In that paper, it is shown that, if we consider the dynamic:

$$\dot{x} = f(x) + u_1 + u_2, \tag{9}$$

then system (7) is weakly hyperbolic and is hyperbolic except some curves on the (p_1, p_2)-plane.

In Section 3, we compare two methods of stating a Pareto optimum for functionals (Y_1, Y_2). In both cases, the referential point is a Nash equilibrium payoff (Y_1^N, Y_2^N), and we require Pareto optimal outcomes to be greater than those for the Nash equilibrium:

$$Y_i^P \geqslant Y_i^N \quad \text{for} \quad i = 1, 2$$

Obviously, the above criterion does not determine Pareto optimal strategies uniquely. Bressan and Shen (2004) receive the uniqueness of Pareto optimal choices by using the following condition:

$$Y_1^P - Y_1^N = Y_2^P - Y_2^N$$

The second criterion of choosing Pareto optimal strategies is based on the concept of the Nash solution to the bargaining problem (see (Nash, 1950)). The pair $(\tilde{Y}_1^P, \tilde{Y}_2^P)$ is such a solution if:

$$(\tilde{Y}_1^P - Y_1^N)(\tilde{Y}_2^P - Y_2^N) \geqslant (Y_1^P - Y_1^N)(Y_2^P - Y_2^N) \qquad \text{for every} \qquad (Y_1^P, Y_2^P).$$

The above condition can be reformulated using function s:

$$s(x, p_1, p_2) = \arg\max_{s>0} \{(Y_1^P(s) - Y_1^N)(Y_2^P(s) - Y_2^N)\}$$

The main advantage of the second approach is that, considering the dynamics such as the Lanchester duopoly model used in (Chintagunta and Vilcassim, 1992) and (Wang and Wu, 2001) in which the dynamic is given by the equation:

$$\dot{x} = u_1(1 - x) - u_2 x \tag{10}$$

and the duopoly model from (Bressan and Shen, 2004) given by the formula:

$$\dot{x} = x(1 - x)(u_1 - u_2) \tag{11}$$

it is possible to compute function s analytically, as we shall study in Section 3, and determine the system (7) effectively. In view of Theorem 2.1, the obtained systems are hyperbolic except for some curves on the (p_1, p_2)-plane.

A natural consequence of the above result should be solving numerically received systems and using them to construct semi-cooperative strategies for empirical examples of a duopoly. Unfortunately, we have no ready algorithms for such problems at the moment. Although, the situation seems not to be hopeless. Hamilton–Jacobi systems can be transformed into systems of conservation laws, and for such problems, there exist numerical algorithms. For now, a numerical solvability will not be the subject of this paper.

The Nash equilibrium is the most common approach to the problem of maximizing payoff (3) for the dynamic given by (1) and (2). As Bressan and Shen (2004) show, in general, such an approach leads to unstable systems of partial differential equations. Therefore, we shall use a Pareto optimality condition. Moreover, our result is not only a theoretical generalization of the Bressan and Shen dynamic, because (9) is not sufficient for empirical research. Let us notice that duopoly models (10) and (11) are not of the (9) form, but they are of the (1) form.

2. PARETO OPTIMAL CHOICES – THE MAIN RESULT

First, we recall the basic definitions and facts concerning the hyperbolicity of linear and nonlinear systems of PDEs. One can find details in (Bressan and Shen, 2004; Serre, 2000).

We consider a linear system on $\mathbb{R}^m$ with constant coefficients:

$$V_t + \sum_{\alpha=1}^{m} A_\alpha V_{x_\alpha} = 0 \tag{12}$$

where t is time, $x \in \mathbb{R}^m$, $V : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^k$. Let us notice that k corresponds to the number of players in a game and m is the dimension of the state space. We define the linear combination:

$$A(\xi) = \sum_{\alpha=1}^{m} \xi_\alpha A_\alpha$$

where $\xi \in \mathbb{R}^m$.

Definition 2.1. *System* (12) *is hyperbolic if there exists a constant* C *such that*

$$\sup_{\xi \in I\mathbb{R}^m} ||\exp iA(\xi)|| \leqslant C$$

where:

$$\exp iA(\xi) = \sum_{n=0}^{\infty} \frac{(iA(\xi))^n}{n!}$$

Definition 2.2. *System* (12) *is weakly hyperbolic, if for every* $\xi \in \mathbb{R}^m$*, the matrix* $A(\xi)$ *has* k *real eigenvalues* $\lambda_1(\xi), \ldots, \lambda_k(\xi)$.

In (Bressan and Shen, 2004), it is shown that the initial value problem for system (12) is well-posed in $L^2(\mathbb{R}^m)$ if and only if the system is hyperbolic. We have the following necessary condition of hyperbolicity.

Lemma 2.1. *If system* (12) *is hyperbolic, then it is weakly hyperbolic.*

The next result refers to the one-dimensional case, when system (12) takes the form:

$$V_t + AV_x = 0 \tag{13}$$

Lemma 2.2. *System* (13) *is hyperbolic if and only if the matrix A admits a basis of real eigenvectors.*

It is also easy to see that the following statement is true.

Remark 1. Let $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$. The matrix A has two real eigenvalues if and only if $(A_{11} - A_{22})^2 + 4A_{12}A_{21} \geqslant 0$. Moreover, if $(A_{11} - A_{22})^2 + 4A_{12}A_{21} > 0$, then the eigenvectors span the space $\mathbb{R}^2$.

In view of Lemma 2.1, it is reasonable to check the weak hyperbolicity in the first place. We mainly receive nonlinear systems, so it is necessary to understand what the hyperbolicity means in this case. Consider the system of Hamilton–Jacobi equations:

$$(V_i)_t + H_i(x, (V_1)_x, \ldots, (V_k)_x) = 0 \qquad i = 1, \ldots, k \tag{14}$$

The linearization of (14) takes the following form:

$$(V_i)_t + \sum_{j,\alpha} \left[\frac{\partial H_i}{\partial p_{j\alpha}}(x, p_1, p_2, \ldots, p_k) \right] \cdot \frac{\partial V_j}{\partial x_\alpha} = 0 \qquad i = 1, \ldots, k \tag{15}$$

where $(x, p_1, p_2, \ldots, p_k) \in \mathbb{R}^{(1+k)m}$ and $p_i = (V_i)_x$. If we denote:

$$(A_\alpha)_{ij} := \frac{\partial H_i}{\partial p_{j\alpha}}(x, p_1, p_2, \ldots, p_k) \tag{16}$$

then equations (15) are of the (12) form.

Definition 2.3. *The nonlinear system* (14) *is hyperbolic (weakly hyperbolic) on the domain $\Omega \in \mathbb{R}^{(1+k)m}$, if for every $(x, p_1, p_2, \ldots, p_k) \in \Omega$ its linearisation* (15) *is hyperbolic (weakly hyperbolic).*

Due to the fact that we are interested in solving empirical problems in a duopoly and applying numerical methods, we need to know that our systems have a unique solution, and this solution's behavior changes continuously with the initial conditions. For this reason, hyperbolicity is crucial.

Our aim is to study the hyperbolicity of a system of Hamilton–Jacobi equations describing value functions generated by a Pareto optimality condition for instantaneous gain functions. The evolution of the state is described by (1) with the initial data given by (2). The goal of the i-th player ($i = 1, 2$) is to maximize his payoff function (3), where g_i and h_i satisfy the assumptions (4), (5). We shall consider instantaneous gain functions:

$$Y_1(x, p_1, p_2, u_1, u_2) = p_1 \cdot (f(x) + \phi(x)u_1 + \psi(x)u_2) - h_1(x, u_1)$$

$$Y_2(x, p_1, p_2, u_1, u_2) = p_2 \cdot (f(x) + \phi(x)u_1 + \psi(x)u_2) - h_2(x, u_2)$$

Fixing $x, p_1, p_2 \in \mathbb{R}^m$ and $s > 0$, we can find Pareto optimal choices $U_i^P(x, p_1, p_2, s)$ for static game $Y_i(x, p_1, p_2, \cdot, \cdot)$, $i = 1, 2$, in the following way: if (u_1^P, u_2^P) is the maximum of function $Y_s = sY_1 + Y_2$, then strategies (u_1^P, u_2^P) give Pareto optimal payoffs in game (Y_1, Y_2). This is the reason why strategies U_i^P depend on s. We choose a smooth function $s(x, p_1, p_2)$ and define the semi-cooperative feedback strategies:

$$U_i^s(x, p_1, p_2) = U_i^P(x, p_1, p_2, s(x, p_1, p_2)) \qquad i = 1, 2 \tag{17}$$

We define the Hamiltonian functions as follows:

$$H_1(x, p_1, p_2) = Y_1(x, p_1, p_2, U_1^s(x, p_1, p_2), U_2^s(x, p_1, p_2))$$

$$H_2(x, p_1, p_2) = Y_2(x, p_1, p_2, U_1^s(x, p_1, p_2), U_2^s(x, p_1, p_2))$$

If value functions:

$$V_i^s(\tau, y) = J_i(\tau, y, U_1^s, U_2^s) \qquad i = 1, 2$$

are smooth, then they satisfy the system of Hamilton–Jacobi equations:

$$\begin{cases} V_{1,t} + H_1(x, \nabla_x V_1, \nabla_x V_2) = 0 \\ V_{2,t} + H_2(x, \nabla_x V_1, \nabla_x V_2) = 0 \end{cases} \tag{18}$$

Theorem 2.1. *Consider problem* (1)–(5). *As gradients* (p_1, p_2) *of the value functions range in open region* $\Omega \subset \mathbb{R}^{2m}$, *assume that the players adopt Pareto optimal strategies of form* (17) *for some smooth function* $s = s(x, p_1, p_2)$. *Then, system* (18) *is weakly hyperbolic on domain* Ω. *Moreover, if we consider one-dimension case* ($m = 1$), *system* (18) *is hyperbolic except for some curves on the* (p_1, p_2)*-plane.*

The method of proof is similar to the proof of Theorem 3 in (Bressan and Shen, 2004).

Proof. We define functions $k_i : \mathbb{R}^m \to \mathbb{R}^n$ – $i = 1, 2$ as follows:

$$k_1(\xi) = k_1(\xi, v, x) = \arg\max_{\omega \in \mathbb{R}^n} \{\xi(f(x) + \phi(x)\omega + \psi(x)v) - h_1(x, \omega)\}$$

and:

$$k_2(\xi) = k_2(\xi, v, x) = \arg\max_{\omega \in \mathbb{R}^n} \{\xi(f(x) + \phi(x)v + \psi(x)\omega) - h_2(x, \omega)\}$$

where $x \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Since h_1, h_2 are smooth functions that satisfy (5), one can observe that

$$\frac{\partial h_1}{\partial u_1}(x, k_1(\xi)) = \xi\phi(x) \qquad \text{and} \qquad \frac{\partial h_2}{\partial u_2}(x, k_2(\xi)) = \xi\psi(x) \tag{19}$$

We seek Pareto optimal choices by maximizing function $Y_s = sY_1 + Y_2$. In view of (19), we can formulate Pareto optimal strategies using functions k_1 and k_2:

$$u_1^P(x, p_1, p_2, s) = k_1(p_1 + \frac{p_2}{s}) \qquad \text{and} \qquad u_2^P(x, p_1, p_2, s) = k_2(sp_1 + p_2) \tag{20}$$

The necessary condition for a local maximum implies that:

$$s\frac{\partial Y_1}{\partial u_1} + \frac{\partial Y_2}{\partial u_1} = s\frac{\partial Y_1}{\partial u_2} + \frac{\partial Y_2}{\partial u_2} = 0 \tag{21}$$

Denoting:

$$Y_i^P = Y_i(x, p_1, p_2, u_1^P(x, p_1, p_2, s), u_2^P(x, p_1, p_2, s)) \qquad i = 1, 2$$

and recalling (21), we obtain the following equality:

$$\frac{\partial Y_1^P}{\partial s} = -\frac{1}{s}\frac{\partial Y_2^P}{\partial s} \tag{22}$$

Now, we compute the linearization of system (18). From (20), we get:

$$Y_1^P = p_1\left(f(x) + \phi(x)k_1\left(p_1 + \frac{p_2}{s}\right) + \psi(x)k_2(sp_1 + p_2)\right) - h_1\left(x, k_1\left(p_1 + \frac{p_2}{s}\right)\right)$$

$$Y_2^P = p_2\left(f(x) + \phi(x)k_1\left(p_1 + \frac{p_2}{s}\right) + \psi(x)k_2(sp_1 + p_2)\right) - h_2(x, k_2(sp_1 + p_2))$$

To clarify further computations, let us temporarily assume that $m = n = 1$ and that $s = \text{const}$. The linearization takes the following form:

$$\begin{bmatrix} f + \phi k_1 + \psi k_2 + p_1(\phi k_1' + s\psi k_2') - h_1'k_1' & p_1(\frac{1}{s}\phi k_1' + \psi k_2') - \frac{1}{s}h_1'k_1' \\ p_2(\phi k_1' + s\psi k_2') - sh_2'k_2' & f + \phi k_1 + \psi k_2 + p_2(\frac{1}{s}\phi k_1' + \psi k_2') - h_2'k_2' \end{bmatrix} \tag{23}$$

where:

$$h_1' = \frac{\partial h_1}{\partial u_1} \qquad \text{and} \qquad h_2' = \frac{\partial h_2}{\partial u_2}$$

Let $a := \psi p_1 k_2' - \frac{1}{s^2}\phi p_2 k_1'$. We can write matrix (23) as follows:

$$A := \begin{bmatrix} f + \phi k_1 + \psi k_2 + sa & a \\ -s^2 a & f + \phi k_1 + \psi k_2 - sa \end{bmatrix} = (f + \phi k_1 + \psi k_2)I + A^\sharp$$

where:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A^\sharp = \begin{bmatrix} sa & a \\ -s^2 a & -sa \end{bmatrix}$$

In view of Remark 1, it is obvious that A is weakly hyperbolic.

Now let $s = s(x, p_1, p_2)$. In this situation, the linearization matrix is the following:

$$A = (f + \phi k_1 + \psi k_2)I + A^\sharp + A^\flat$$

where:

$$A^\flat = \begin{bmatrix} \frac{\partial Y_1^P}{\partial s}\frac{\partial s}{\partial p_1} & \frac{\partial Y_1^P}{\partial s}\frac{\partial s}{\partial p_2} \\ \frac{\partial Y_2^P}{\partial s}\frac{\partial s}{\partial p_1} & \frac{\partial Y_2^P}{\partial s}\frac{\partial s}{\partial p_2} \end{bmatrix}$$

Using (22) and denoting $c := \frac{\partial Y_1^P}{\partial s}\frac{\partial s}{\partial p_1}$, $d := \frac{\partial Y_1^P}{\partial s}\frac{\partial s}{\partial p_2}$, we obtain:

$$A^\flat = \begin{bmatrix} c & d \\ -sc & -sd \end{bmatrix}$$

From Remark 1, it is easy to verify that matrix A has two real eigenvalues: $\lambda_1 = f + \phi k_1 + \psi k_2$ and $\lambda_2 = f + \phi k_1 + \psi k_2 + c - sd$; thus, the system is weakly hyperbolic. Furthermore, the system is hyperbolic when $c \neq sd$ and $p_i \neq 0$ for $i = 1, 2$.

Now let $m, n \in \mathbb{N}$. We need to verify if matrix:

$$A(\xi) = \sum_{\alpha=1}^{m} \xi_\alpha A_\alpha$$

is weakly hyperbolic where, as in (16),

$$A_\alpha = \left[\frac{\partial H_i}{\partial p_{j\alpha}}(x, p_1, p_2)\right]_{i,j=1}^{2}$$

Repeating the reasoning for α's coordinate of p_1 and p_2, we receive:

$$A_\alpha = (f_\alpha + (\phi k_1)_\alpha + (\psi k_2)_\alpha)I + A^\sharp_\alpha + A^\flat_\alpha$$

where:

$$A^\sharp_\alpha = \begin{bmatrix} sa_\alpha & a_\alpha \\ -sa_\alpha & -sa_\alpha \end{bmatrix}, \qquad A^\flat_\alpha = \begin{bmatrix} c_\alpha & d_\alpha \\ -sc_\alpha & -sd_\alpha \end{bmatrix}$$

and $a_\alpha = \psi(Dk_2 \cdot p_1)_\alpha - \frac{1}{s^2}\phi(Dk_1 \cdot p_2)_\alpha$, $c_\alpha = \frac{\partial Y_1^P}{\partial s}\frac{\partial s}{\partial p_{1\alpha}}$, $d_\alpha = \frac{\partial Y_1^P}{\partial s}\frac{\partial s}{\partial p_{2\alpha}}$. This means that matrix $A(\xi)$ has the following form:

$$A(\xi) = \sum_{\alpha=1}^{m} \xi_\alpha A_\alpha = (\xi \cdot f + \xi \cdot \phi k_1 + \xi \cdot \psi k_2)I + A^\sharp(\xi) + A^\flat(\xi)$$

where:

$$A^\sharp(\xi) = \begin{bmatrix} \xi \cdot sa & \xi \cdot a \\ -\xi \cdot s^2 a & -\xi \cdot sa \end{bmatrix}, \qquad A^\flat(\xi) = \begin{bmatrix} \xi \cdot c & \xi \cdot d \\ -\xi \cdot sc & -\xi \cdot sd \end{bmatrix}$$

Matrix $A(\xi)$ has the two real eigenvalues:

$\lambda_1(\xi) = \xi \cdot (f + \phi k_1 + \psi k_2)$ and $\lambda_2(\xi) = \xi \cdot (f + \phi k_1 + \psi k_2 + c - sd)$. ■

Remark 2. If s is constant, then our problem becomes a cooperative game, and there is no guarantee that Pareto optimal payoffs dominate Nash payoffs. Such dominance is crucial for our considerations, because we want to improve outcomes in a reasonable way.

3. THE UNIQUENESS OF THE PARETO OPTIMAL CHOICES

The choice of Pareto optimal strategies is a very important issue. Since Pareto optimal outcomes are not unique, we present two meaningfully different criteria. In this section, we compare the Bressan and Shen criterion (2004) with the criterion proposed by us, which is based on the Nash solution to the bargaining problem. Finally, we determine Pareto optimal solutions for two duopoly models.

Bressan and Shen formulate the choice of s basing on the fairness conditions:

$$Y_i^P(s) > Y_i^N \qquad \text{for} \qquad i = 1,2 \tag{24}$$

and:

$$Y_1^P(s) - Y_1^N = Y_2^P(s) - Y_2^N \tag{25}$$

Condition (24) is necessary to receive better outcomes than the Nash equilibrium ones, and it is essential to convince players to use a Pareto optimal approach. Unfortunately, conditions (24), (25) are not easy to apply in the examples. Accordingly, we suggest using the Nash solution to the bargaining problem. Firstly, the choice should not make the payoffs worse:

$$Y_i^P(s) \geqslant Y_i^N \qquad \text{for} \qquad i = 1,2 \tag{26}$$

Pair $(\tilde{Y}_1^P, \tilde{Y}_2^P)$ is the Nash solution to the bargaining problem if:

$$(\tilde{Y}_1^P - Y_1^N)(\tilde{Y}_2^P - Y_2^N) \geqslant (Y_1^P - Y_1^N)(Y_2^P - Y_2^N) \qquad \text{for every} \qquad (Y_1^P, Y_2^P)$$

In the examples, we use the following reformulated form:

$$s(x, p_1, p_2) = \arg\max_{s>0}\{(Y_1^P(s) - Y_1^N)(Y_2^P(s) - Y_2^N)\} \tag{27}$$

If the intersection of the image of function $Y = (Y_1,\ Y_2)$ and set $\{(y_1,\ y_2) : y_i \geqslant Y_i^N,$ $i = 1,2\}$ is convex, then conditions (26), (27) provide the unique s. We compare these two approaches for two dynamics, the Lanchester duopoly model:

$$\dot{x} = u_1(1-x) - u_2 x \tag{28}$$

and the duopoly model from (Bressan and Shen, 2004):

$$\dot{x} = x(1-x)(u_1 - u_2) \tag{29}$$

In both cases, state $x \in [0,1]$ characterizes the market share. We shall use the following payoff function:

$$J_i(\tau, y, u_1, u_2) = x_i(T) + \int_\tau^T \left[x_i(t) - \frac{1}{2}u_i^2(t)\right] dt \tag{30}$$

where $x_1(t) = x(t)$ is the market share of the first company at time $t \in [\tau, T]$, while $x_2(t) = 1 - x(t)$ is the market share of the second. Both methods require comparing new values with Nash equilibrium payoffs Y_1^N, Y_2^N – the instantaneous gain functions for Nash equilibrium strategies u_1^N, u_2^N. The strategies can be found from the following conditions:

$$\frac{\partial Y_1}{\partial u_1} = 0 \qquad \text{and} \qquad \frac{\partial Y_2}{\partial u_2} = 0 \tag{31}$$

Example 1

Let us consider the Lanchester duopoly model, which is given by (28), with the payoff function (30). The instantaneous gain functions take form:

$$Y_i(x, p_1, p_2, u_1, u_2) = p_i(u_1(1-x) - u_2 x) + x_i - \frac{1}{2}u_i^2 \qquad i = 1, 2$$

where $x_1 = x$ and $x_2 = 1 - x$. Using (31), we obtain that $u_1^N = p_1(1-x)$ and $u_2^N = -p_2 x$. The Nash payoffs are the following:

$$Y_1^N = Y_1(x, p_1, p_2, u_1^N, u_2^N) = \frac{1}{2}p_1^2(1-x)^2 + p_1 p_2 x^2 + x$$

$$Y_2^N = Y_2(x, p_1, p_2, u_1^N, u_2^N) = \frac{1}{2}p_2^2 x^2 + p_1 p_2 (1-x)^2 + 1 - x$$

Now, we find the set of Pareto optimal choices. We maximize function $Y_s = sY_1 + Y_2$:

$$Y_s(x, p_1, p_2, u_1, u_2) = sp_1(u_1(1-x) - u_2 x) + sx - s\frac{1}{2}u_1^2 + p_2(u_1(1-x) - u_2 x) + 1 - x - \frac{1}{2}u_2^2.$$

Using necessary condition:

$$\frac{\partial Y_s}{\partial u_1} = 0 \qquad \text{and} \qquad \frac{\partial Y_s}{\partial u_2} = 0 \tag{32}$$

we receive $u_1^P = (p_1 + \frac{1}{s}p_2)(1-x)$ and $u_2^P = -(sp_1 + p_2)x$. The respective payoffs are:

$$Y_1^P = p_1\left(\left(p_1 + \frac{1}{s}p_2\right)(1-x)^2 + (sp_1 + p_2)x^2\right) + x - \frac{1}{2}\left(p_1 + \frac{1}{s}p_2\right)^2 (1-x)^2$$

$$Y_2^P = p_2\left(\left(p_1 + \frac{1}{s}p_2\right)(1-x)^2 + (sp_1 + p_2)x^2\right) + 1 - x - \frac{1}{2}\left(sp_1 + p_2\right)^2 x^2$$

Firstly, we shall use fairness condition (24) and (25). Condition (25) allows us to present function s as one of the solutions of the following equation:

$$(p_1 x)^2 s^4 + 2(p_1 x)^2 s^3 - 2(p_2(1-x))^2 s - (p_2(1-x))^2 = 0 \tag{33}$$

Unfortunately, condition (33) does not provide solutions that could be presented in one simple, analytically computed formula. On the other hand, applying conditions (26) and (27) we obtain that the seeking function s is given by the formula:

$$s(x, p_1, p_2) = \left(\frac{p_2(1-x)}{p_1 x}\right)^{\frac{2}{3}} \tag{34}$$

The Hamilton functions for (34) are the following:

$$H_1(x,p_1,p_2)=\frac{1}{2}\left(p_1^2-\left(\left(\frac{x}{1-x}\right)^2 p_1^2p_2\right)^{\frac{2}{3}}\right)(1-x)^2+\left(p_1p_2+\left(\left(\frac{1-x}{x}\right)p_1^2p_2\right)^{\frac{2}{3}}\right)x^2+x$$

$$H_2(x,p_1,p_2)=\frac{1}{2}\left(p_2^2-\left(\left(\frac{1-x}{x}\right)^2 p_1p_2^2\right)^{\frac{2}{3}}\right)x^2+\left(p_1p_2+\left(\left(\frac{x}{1-x}\right)p_1p_2^2\right)^{\frac{2}{3}}\right)(1-x)^2+1-x$$

The matrix of the linearization of (18), in this case, takes the form:

$$\begin{bmatrix} \left(p_1-\frac{2\phi^2(x)\left(p_1p_2^2\right)^{\frac{1}{3}}}{3}\right)(1-x)^2 & -\frac{\phi^2(x)\left(p_1^4p_2^{-1}\right)^{\frac{1}{3}}}{3}(1-x)^2 \\ +\left(p_2+\frac{4\left(p_1p_2^2\right)^{\frac{1}{3}}}{3\phi(x)}\right)x^2 & +\left(p_1+\frac{2\left(p_1^4p_2^{-1}\right)^{\frac{1}{3}}}{3\phi(x)}\right)x^2 \\ & \\ -\frac{\left(p_1^{-1}p_2^4\right)^{\frac{1}{3}}}{3\phi^2(x)}x^2 & \left(p_2-\frac{2\left(p_1^2p_2\right)^{\frac{1}{3}}}{3\phi^2(x)}\right)x^2 \\ +\left(p_2+\frac{2\phi(x)\left(p_1^{-1}p_2^4\right)^{\frac{1}{3}}}{3}\right)(1-x)^2 & +\left(p_1+\frac{4\phi(x)\left(p_1^2p_2\right)^{\frac{1}{3}}}{3}\right)(1-x)^2 \end{bmatrix}_{p_1=\nabla_x V_1,\, p_2=\nabla_x V_2}$$

where $\phi(x)=\left(\frac{x}{1-x}\right)^{\frac{2}{3}}$.

Example 2

Let us consider the second duopoly model, which is given in (29). The payoff functions are given in (30); thus, the instantaneous gain functions take the following form:

$$Y_i(x,p_1,p_2,u_1,u_2)=p_ix(1-x)(u_1-u_2)+x_i-\frac{1}{2}u_i^2 \qquad i=1,2$$

where $x_1=x$ and $x_2=1-x$. Using (31), we obtain that $u_1^N=p_1x(1-x)$ and $u_2^N=-p_2x(1-x)$ with the Nash payoffs:

$$Y_1^N=(x(1-x))^2\left(\frac{1}{2}p_1^2+p_1p_2\right)+x, \qquad Y_2^N=(x(1-x))^2\left(\frac{1}{2}p_2^2+p_1p_2\right)+1-x$$

Now, we need to find the set of Pareto optimal choices. To do that, we shall maximize function $Y_s=sY_1+Y_2$, ($s>0$ is fixed):

$$Y_s(x,p_1,p_2,u_1,u_2)=sp_1x(1-x)(u_1-u_2)+sx-s\frac{1}{2}u_1^2+p_2x(1-x)(u_1-u_2)+1-x-\frac{1}{2}u_2^2$$

Using necessary condition (32), we get $u_1^P=\left(p_1+\frac{1}{s}p_2\right)x(1-x)$ and $u_2^P=-(sp_1+p_2)$ $x(1-x)$, and the Pareto optimal payoffs are:

$$Y_1^P=(x(1-x))^2\left[\left(\frac{1}{2}+s\right)p_1^2+p_1p_2-\frac{p_2^2}{2s^2}\right]+x$$

$$Y_2^P = (x(1-x))^2 \left[\left(\frac{1}{2} + \frac{1}{s} \right) p_2^2 + p_1 p_2 - \frac{s^2 p_1^2}{2} \right] + 1 - x$$

Applying (25), we get a very similar polynomial as in Example 1:

$$(p_1)^2 s^4 + 2(p_1)^2 s^3 - 2(p_2)^2 s - (p_2)^2 = 0$$

On the other hand, from conditions (26) and (27), we find that seeking function s is given by the following formula:

$$s(x, p_1, p_2) = \left(\frac{p_2}{p_1} \right)^{\frac{2}{3}} \tag{35}$$

The Hamilton functions for (35) are the following:

$$H_1(x, p_1, p_2) = x^2(1-x)^2 \left(\frac{1}{2} p_1^2 + p_1 p_2 + \frac{1}{2} \left(p_1^2 p_2 \right)^{\frac{2}{3}} \right) + x$$

$$H_2(x, p_1, p_2) = x^2(1-x)^2 \left(\frac{1}{2} p_2^2 + p_1 p_2 + \frac{1}{2} \left(p_1 p_2^2 \right)^{\frac{2}{3}} \right) + 1 - x$$

The linearization of the system takes the following form:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_t + \begin{bmatrix} \phi(x) \left(p_1 + p_2 + \frac{2}{3} \left(p_1 p_2^2 \right)^{\frac{1}{3}} \right) & \phi(x) \left(p_1 + \frac{1}{3} \left(p_1^4 p_2^{-1} \right)^{\frac{1}{3}} \right) \\ \phi(x) \left(p_2 + \frac{1}{3} \left(p_1^{-1} p_2^4 \right)^{\frac{1}{3}} \right) & \phi(x) \left(p_1 + p_2 + \frac{2}{3} \left(p_1^2 p_2 \right)^{\frac{1}{3}} \right) \end{bmatrix}_{p_1 = \nabla_x V_1, p_2 = \nabla_x V_2} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $\phi(x) = x^2(1-x)^2$.

Criteria (26) and (27) provide an effective analytical formula describing Hamilotnian functions H_1, H_2 in system (18) in both duopoly models. Our next aim is to solve the received systems numerically and to compare the obtained solutions with empirical data.

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