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Three-Machine Flowshop Scheduling Problem to Minimize Total Completion Time with Bounded Setup and Processing Times

Ali Allahverdi*

Abstract. The three-machine flowshop scheduling problem to minimize total completion time is studied where setup times are treated as separate from processing times. Setup and processing times of all jobs on all machines are unknown variables before the actual occurrence of these times. The lower and upper bounds for setup and processing times of each job on each machine is the only information that is available. In such a scheduling environment, there may not exist a unique schedule that remains optimal for all possible realizations of setup and processing times. Therefore, it is desired to obtain a set of dominating schedules (which dominate all other schedules) if possible. The objective for such a scheduling environment is to reduce the size of dominating schedule set. We obtain global and local dominance relations for a three-machine flowshop scheduling problem. Furthermore, we illustrate the use of dominance relations by numerical examples and conduct computational experiments on randomly generated problems to measure the effectiveness of the developed dominance relations. The computational experiments show that the developed dominance relations are quite helpful in reducing the size of dominating schedules.

Keywords: scheduling, flowshop, dominance relations, bounded processing and setup times.

Mathematics Subject Classification: 90B30, 90B36.

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1. INTRODUCTION

The first scientific work on flowshop scheduling problems was conducted by Johnson (1954). Since then, the flowshop scheduling problem has attracted considerable attention from researchers and hundreds of papers have been published in scheduling related journals. The vast majority of research on the problem assumes that job processing times are known fixed values in advance. In other words, the precise information about how long each job will take on each machine is available. It is true that there are many problems in real life where job processing times can be modeled as known fixed values. On the other hand, it is not realistic to assume they are known fixed values for some other scheduling problems. For such scheduling environments, job processing

* Department of Industrial and Management Systems Engineering, College of Engineering and Petroleum, Kuwait University, Kuwait. E-mail: allahverdi@kuc01.kuniv.edu.kw.

times are unknown variables and the only information that can be obtained is about lower and upper bounds for each job, which may be called, bounded processing times. The two machine flowshop scheduling problem with bounded processing times was addressed by Allahverdi and Sotskov (2003) to minimize minimize makespan. The same problem but with total completion time criterion was studied by Sotskov et al. (2004). Setup times were ignored by both Allahverdi and Sotskov (2003), and Sotskov et al. (2004).

The assumption of including setup times in processing times is a common assumption in the flowshop scheduling research. While this assumption may be justified for some real scheduling problems, other situations call for explicit setup time consideration. For example, the production of seamless steel tube in iron and steel industries (Tang and Huang, 2005) or group scheduling in flexible flowshops (Logendran et al., 2005). The practical situations in which setup times must be considered as separate include chemical, pharmaceutical, printing, food processing, metal processing, and semiconductor industries, see Allahverdi et al. (1999, 2007) for surveys on scheduling problems with separate setup times. The performance measure may be improved by considering setup times as separate from processing times.

Some researchers addressed the flowshop problem with setup times. Bagga and Khurana (1986) and Allahverdi (2000) addressed the two-machine separate setup time problem with respect to total completion time criterion. Allahverdi and Al-Anzi (2006) studied the three-machine flowshop problem with total completion time criterion where setup times are treated separately. In the aforementioned three studies, the setup times are considered as separate from processing times but assumed to be *deterministic,* that is, known before scheduling and fixed during a realization of the process. In reality, this assumption may not be valid for some environments, and therefore, setup times have to be considered as random variables. Kim and Bobrowski (1997) pointed out that in many real-world situations, setup times vary stochastically as a result of random factors such as: crew skills; temporary shortage of equipment, tools and setup crews; and unexpected breakdown of fixtures and tools during setup operations. They stated that assuming these random setup times to be fixed may lead to development of inefficient results. Moreover, it is sometimes difficult to find an appropriate probability distribution for random setup times. Allahverdi et al. (2003) considered the two-machine flowshop scheduling problem to minimize total completion time with bounded setup times but where job processing times are assumed to be known fixed values. The only research that we are aware of where both processing times and setup times are bounded and the criterion is total completion time is by Allahverdi (2006) who addresses the two-machine flowshop scheduling problem. In this paper, the results of Allahverdi (2006) are extended to the three-machine flowshop problem.

In this paper, we study the three-machine flowshop scheduling problem to minimize total completion time with unknown setup and unknown processing times where only lower and upper bounds for each setup and processing times of each job are known before scheduling takes place. It has been observed that although the exact values of setup and processing times may not be known before scheduling, some upper and lower bounds on job setup and processing times are easy to obtain in most practical cases. This information on the bounds of setup and processing times is important, and hence, it should be utilized in finding a solution for the scheduling problem.

The rest of this paper is organized as follows. Problem description is provided in Section 2 and the formulation is given in Section 3. Global and local dominance relations are established in Sections 4. A numerical example is provided in Section 5. Computational analysis of the global and local dominance relations is conducted in Section 6 and concluding remarks are made in Section 7.

2. PROBLEM DESCRIPTION

We consider a three-machine flowshop scheduling problem in which job processing times are unknown variables where only a lower bound $LBt_{j,m} \geq 0$ and an upper bound $UBt_{j,m} \geq LBt_{j,m}$ of the processing time $t_{j,m}$ of job j on machine m are given. The exact value of a job processing time will be only known once the job is completed. Although this exact value of the job processing time may not be known before realization or execution, it is known that it will have a value between the lower and upper bounds. Similarly, before jobs are completed, setup times are also unknown variables with a lower bound $LBs_{j,m} \geq 0$ and an upper bound $UBs_{j,m} \geq LBs_{j,m}$ of the setup time $s_{j,m}$ of job j on machine m. We may represent this flowshop problem by $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$. $F3$ denotes that it is a three-machine flowshop. $LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}$ and $LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}$ show that both processing and setup times are unknown variables with some lower and upper bounds. The last term $\sum C_j$ indicates that the objective of the problem is to minimize the total completion time. The problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$ may be considered as a stochastic flowshop problem under uncertainty of setup and processing times when there is no prior information about probability distributions of the random setup and processing times. It is only known that setup and processing times of each job will fall between some given lower and upper bounds with probability one. Similar problems have been addressed in the literature for the case where job processing times are random variables but setup times are assumed to be zero, see Lai et al. (1997), Lai and Sotskov (1999), Allahverdi and Sotskov (2003), and Sotskov et al. (2004). Also Allahverdi et al. (2003) addressed the problem for the case where job processing times are fixed values but setup times are separate and random variables with lower and upper bounds. Allahverdi (2006) addressed the two-machine flowshop scheduling problem where both processing times and setup times are random variables with lower and upper bounds, i.e., he addressed the same problem that is addressed in this paper but only for the two-machine case. The current paper extends the results of Allahverdi (2006) to the three-machine case.

It should be noted that the $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$ problem is NP-hard since it is known that the $F3|LBt_{j,m} = t_{j,m} = UBt_{j,m}; LBs_{j,m} = s_{j,m} = UBs_{j,m}|\sum C_j$ problem is NP-hard. It should be also noted that the objective function value depends on the realization of setup and processing times. We only consider the set of permutation schedules, and there are $n!$

sequences (permutations) $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_{n!}\}$ for the problem of $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$ that will be considered in finding the optimal solution.

3. FORMULATION

Let:

$t_{j,k}$ — the processing time of job j *(j = 1,2,...,n)* on machine *k(k =1,2,3)*,
$s_{j,k}$ — the setup time of job j on machine k, (setup times are assumed to be sequence independent)
$C_{j,k}$ — the completion time of job j on machine k,
TCT — total completion time.

Also let [j, k] denote the job in position j on machine k. Therefore, $C_{[j,k]}$ denotes the completion time of the job in position j on machine k. $t_{[j,k]}$ and $s_{[j,k]}$ are defined similarly.

Let $ST_{[j,k]}$ denote the sum of the setup and processing times of jobs in positions 1, 2, ..., j on machine k, i.e.,

$$ST_{[j,k]} = \sum_{r=1}^{j}(s_{[r,k]} + t_{[r,k]}), j = 1, 2, \ldots, n \quad \text{and} \quad k = 1, 2, 3$$

Let

$$\delta_{[j]} = ST_{[j,1]} - (ST_{[j-1,2]} + s_{[j,2]}), \quad j = 1, 2, \ldots, n$$

where $ST_{[0,2]} = 0$. Let $IT_{[j,2]}$ denote total idle time on the second machine until the job in position j on the machine is completed. It can be shown that (see Allahverdi, 2000)

$$IT_{[j,2]} = \max\{0, \delta_{[1]}, \delta_{[2]}, \ldots, \delta_{[j]}\} \tag{1}$$

Therefore,

$$C_{[j,2]} = ST_{[j,2]} + IT_{[j,2]}.$$

Let

$$\phi_{[j]} = ST_{[j,2]} + IT_{[j,2]} - (ST_{[j-1,3]} + s_{[j,3]}), \quad j = 1, 2, \ldots, n$$

where $ST_{[0,3]} = 0$. If $IT_{[j,3]}$ denotes the total idle time on the third machine until the job in position j on the machine is completed, then it can be shown that (Allahverdi and Al-Anzi, 2006)

$$IT_{[j,3]} = max\{0, \phi_{[1]}, \phi_{[2]}, \ldots, \phi_{[j]}\} \tag{2}$$

Hence,

$$C_{[j,3]} = ST_{[j,3]} + IT_{[j,3]}.$$

Once the completion times of jobs on the last (third) machine are known, then,

$$TCT = \sum_{j=1}^{n} (ST_{[j,3]} + IT_{[j,3]}) \tag{3}$$

It should be noted that throughout this paper we only consider permutation flowshops.

For each job $j \in J$ and machine $m \in M$, any feasible realization $t_{j,m}$ of processing times satisfies the inequalities: $LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}$. Similarly, any feasible realization $s_{j,m}$ of setup time satisfies the inequalities: $LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}$. Before jobs are completed, we only know the lower and upper bounds of processing and setup times for the jobs given by the above inequalities, which define polytope PT of feasible vectors $t = (t_{1,1}, t_{1,2}, t_{1,3}, t_{2,1}, t_{2,2}, t_{2,3}, \ldots, t_{n,1}, t_{n,2}, t_{n,3})$ and $s = (s_{1,1}, s_{1,2}, s_{1,3}, s_{2,1}, s_{2,2}, s_{2,3}, \ldots, s_{n,1}, s_{n,2}, s_{n,3})$ of processing and setup times as follows: $PT = \{t : LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}$ and $s : LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}$, $j \in J, m \in M\}$.

We use the following definition of a solution to the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}| \sum C_j$. A set of sequences $\Phi^* \subseteq \Phi$ is a *solution* to the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}| \sum C_j$ if for each feasible vector $t \in PT$ of processing times and each feasible vector $s \in PT$ of setup times, the set Φ^* contains at least one optimal sequence. Thus, the whole set Φ of sequences is a trivial solution for the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}| \sum C_j$. However, it is only possible to construct the whole set Φ for a small number of jobs. It is also impractical to choose the best sequence from a large set Φ^* of candidates as the processing and setup times of jobs evolves. Therefore, it is important to minimize the cardinality of solution Φ^* constructed for the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}| \sum C_j$. We introduce the following dominance relations on the set of sequences Φ.

For the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}| \sum C_j$ a sequence $\Phi_1 \in \Phi$ dominates a sequence $\Phi_2 \in \Phi$ with respect to PT if the inequality $TCT(\Phi_1) \leq TCT(\Phi_2)$ holds for any vectors $t \in PT$ and $s \in PT$.

By the aforementioned definition, a set of sequences $\Phi^* \subseteq \Phi$ is a solution to the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}| \sum C_j$ if for each sequence $\Phi_k \in \Phi$, there exists a sequence from the set Φ^* that dominates the sequence Φ_k with respect to PT.

4. DOMINANCE RELATIONS

Let φ_h denote a subsequence of a complete sequence $\Phi_u \in \Phi$ of all the n jobs. Therefore, the notations of $\Phi_1 = (\varphi_1, d, \varphi_2, k, \varphi_3)$ and $\Phi_2 = (\varphi_1, k, \varphi_2, d, \varphi_3)$ mean that the two sequences of $\Phi_1 \in \Phi$ and $\Phi_2 \in \Phi$ have the same jobs in all positions except that the jobs $d \in J$ and $k \in J$ are interchanged. When the jobs d and k are adjacent, such two complete sequences of $\Phi_3 \in \Phi$ and $\Phi_4 \in \Phi$ can be expressed as follows $\Phi_3 = (\varphi_1, d, k, \varphi_2)$ and $\Phi_4 = (\varphi_1, k, d, \varphi_2)$.

Theorem 1. *For the problem* $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$, *the sequence* $\Phi_1 = (\phi_1, j, \phi_2, i, \phi_3) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, \phi_2, j, \phi_3) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $UBs_{j,1} + UBt_{j,1} + UBs_{i,2} \leq LBs_{i,1} + LBt_{i,1} + LBs_{j,2}$,
(ii) $UBt_{i,2} \leq LBt_{j,2}$,
(iii) $UBs_{j,2} + UBt_{j,2} + UBs_{i,3} \leq LBs_{i,2} + LBt_{i,2} + LBs_{j,3}$,
(iv) $UBt_{i,3} \leq LBt_{j,3}$, *and*
(v) $UBs_{j,3} + UBt_{j,3} \leq LBs_{i,3} + LBt_{i,3}$.

Proof. Consider exchanging the positions of two jobs on a three-machine flowshop in a sequence π_1 that has job i in an arbitrary position g and job j in position h. Consider another sequence that is obtained from the sequence π_1 by only interchanging jobs i and j. Call the sequence obtained from π_1 as π_2, i.e., $\pi_1 = \ldots, i, \ldots, j, \ldots$ and $\pi_2 = \ldots, j, \ldots, i, \ldots$. If it is shown that $TCT(\pi_2) \leq TCT(\pi_1)$, then, sequence π_2 would be no worse than sequence π_1, and, therefore, job j precedes job i in a sequence that minimizes total completion time.

For the job in position g,

$$\delta_{[g]}(\pi_1) = ST_{[g-1,1]}(\pi_1) + s_{i,1} + t_{i,1} - ST_{[g-1,2]}(\pi_1) - s_{i,2}, \tag{4}$$

$$\delta_{[g]}(\pi_2) = ST_{[g-1,1]}(\pi_2) + s_{j,1} + t_{j,1} - ST_{[g-1,2]}(\pi_2) - s_{j,2}, \tag{5}$$

$$\begin{aligned}\phi_{[g]}(\pi_1) = {} & ST_{[g-1,2]}(\pi_1) + s_{i,2} + t_{i,2} + \max\{IT_{[g-1,2]}(\pi_1), \delta_{[g]}(\pi_1)\} \\ & - ST_{[g-1,3]}(\pi_1) - s_{i,3}, \quad \text{and}\end{aligned} \tag{6}$$

$$\begin{aligned}\phi_{[g]}(\pi_2) = {} & ST_{[g-1,2]}(\pi_2) + s_{j,2} + t_{j,2} + \max\{IT_{[g-1,2]}(\pi_2), \delta_{[g]}(\pi_2)\} \\ & - ST_{[g-1,3]}(\pi_2) - s_{j,3}.\end{aligned} \tag{7}$$

Moreover, for *r=g+1, g+2, ..., h-1*,

$$\begin{aligned}\delta_{[r]}(\pi_1) = {} & ST_{[g-1,1]}(\pi_1) + s_{i,1} + t_{i,1} + \sum_{d=g+1}^{r} (s_{[d,1]} + t_{[d,1]}) \\ & - ST_{[g-1,2]}(\pi_1) - s_{i,2} - t_{i,2} - \sum_{d=g+1}^{r-1} (s_{[d,2]} + t_{[d,2]}) - s_{[r,2]},\end{aligned} \tag{8}$$

$$\begin{aligned}\delta_{[r]}(\pi_2) = {} & ST_{[g-1,1]}(\pi_2) + s_{j,1} + t_{j,1} + \sum_{d=g+1}^{r} (s_{[d,1]} + t_{[d,1]}) \\ & - ST_{[g-1,2]}(\pi_2) - s_{j,2} - t_{j,2} - \sum_{d=g+1}^{r-1} (s_{[d,2]} + t_{[d,2]}) - s_{[r,2]},\end{aligned} \tag{9}$$

$$\begin{aligned}\phi_{[r]}(\pi_1) = ST_{[g-1,2]}(\pi_1) + s_{i,2} + t_{i,2} + \sum_{d=g+1}^{r}(s_{[d,2]} + t_{[d,2]}) \\ + \max\{IT_{[g-1,2]}(\pi_1), \delta_{[g]}(\pi_1), \delta_{[g+1]}(\pi_1), \ldots, \delta_{[r]}(\pi_1)\} \\ - ST_{[g-1,3]}(\pi_1) - s_{i,3} - t_{i,3} - \sum_{d=g+1}^{r-1}(s_{[d,3]} + t_{[d,3]}) - s_{[r,3]},\end{aligned} \tag{10}$$

$$\begin{aligned}\phi_{[r]}(\pi_2) = ST_{[g-1,2]}(\pi_2) + s_{j,2} + t_{j,2} + \sum_{d=g+1}^{r}(s_{[d,2]} + t_{[d,2]}) \\ + \max\{IT_{[g-1,2]}(\pi_2), \delta_{[g]}(\pi_2), \delta_{[g+1]}(\pi_2), \ldots, \delta_{[r]}(\pi_2)\} \\ - ST_{[g-1,3]}(\pi_2) - s_{j,3} - t_{j,3} - \sum_{d=g+1}^{r-1}(s_{[d,3]} + t_{[d,3]}) - s_{[r,3]}.\end{aligned} \tag{11}$$

On the other hand, for the job in position h,

$$\begin{aligned}\delta_{[h]}(\pi_1) = ST_{[g-1,1]}(\pi_1) + s_{i,1} + t_{i,1} + \sum_{d=g+1}^{h-1}(s_{[d,1]} + t_{[d,1]}) + s_{j,1} + t_{j,1} \\ - ST_{[g-1,2]}(\pi_1) - s_{i,2} - t_{i,2} - \sum_{d=g+1}^{h-1}(s_{[d,2]} + t_{[d,2]}) - s_{j,2},\end{aligned} \tag{12}$$

$$\begin{aligned}\delta_{[h]}(\pi_2) = ST_{[g-1,1]}(\pi_2) + s_{j,1} + t_{j,1} + \sum_{d=g+1}^{h-1}(s_{[d,1]} + t_{[d,1]}) + s_{i,1} + t_{i,1} \\ - ST_{[g-1,2]}(\pi_2) - s_{j,2} - t_{j,2} - \sum_{d=g+1}^{h-1}(s_{[d,2]} + t_{[d,2]}) - s_{i,2},\end{aligned} \tag{13}$$

$$\begin{aligned}\phi_{[h]}(\pi_1) = ST_{[g-1,2]}(\pi_1) + s_{i,2} + t_{i,2} + \sum_{d=g+1}^{h-1}(s_{[d,2]} + t_{[d,2]}) + s_{j,2} + t_{j,2} \\ + max\{IT_{[g-1,2]}(\pi_1), \delta_{[g]}(\pi_1), \delta_{[g+1]}(\pi_1), \ldots, \delta_{[h]}(\pi_1)\} \\ - ST_{[g-1,3]}(\pi_1) - s_{i,3} - t_{i,3} - \sum_{d=g+1}^{h-1}(s_{[d,3]} + t_{[d,3]}) - s_{j,3},\end{aligned} \tag{14}$$

$$\begin{aligned}\phi_{[h]}(\pi_2) = ST_{[g-1,2]}(\pi_2) + s_{j,2} + t_{j,2} + \sum_{d=g+1}^{h-1}(s_{[d,2]} + t_{[d,2]}) + s_{i,2} + t_{i,2} \\ + max\{IT_{[g-1,2]}(\pi_2), \delta_{[g]}(\pi_2), \delta_{[g+1]}(\pi_2), \ldots, \delta_{[h]}(\pi_2)\} \\ - ST_{[g-1,3]}(\pi_2) - s_{j,3} - t_{j,3} - \sum_{d=g+1}^{h-1}(s_{[d,3]} + t_{[d,3]}) - s_{i,3}.\end{aligned} \tag{15}$$

Finally, for $r = h+1, h+2, \ldots, n$,

$$\begin{aligned}
\delta_{[r]}(\pi_1) = ST_{[g-1,1]}(\pi_1) &+ s_{i,1} + t_{i,1} \\
&+ \sum_{d=g+1}^{h-1} (s_{[d,1]} + t_{[d,1]}) + s_{j,1} + t_{j,1} \\
&+ \sum_{d=h+1}^{r} (s_{[d,1]} + t_{[d,1]}) - ST_{[g-1,2]}(\pi_1) - s_{i,2} - t_{i,2} \\
&- \sum_{d=g+1}^{h-1} (s_{[d,2]} + t_{[d,2]}) - s_{j,2} - t_{j,2} \\
&- \sum_{d=h+1}^{r-1} (s_{[d,2]} + t_{[d,2]}) - s_{[r,2]},
\end{aligned} \tag{16}$$

$$\begin{aligned}
\delta_{[r]}(\pi_2) = ST_{[g-1,1]}(\pi_2) &+ s_{j,1} + t_{j,1} \\
&+ \sum_{d=g+1}^{h-1} (s_{[d,1]} + t_{[d,1]}) + s_{i,1} + t_{i,1} \\
&+ \sum_{d=h+1}^{r} (s_{[d,1]} + t_{[d,1]}) - ST_{[g-1,2]}(\pi_2) - s_{j,2} - t_{j,2} \\
&- \sum_{d=g+1}^{h-1} (s_{[d,2]} + t_{[d,2]}) - s_{i,2} - t_{i,2} \\
&- \sum_{d=h+1}^{r-1} (s_{[d,2]} + t_{[d,2]}) - s_{[r,2]},
\end{aligned} \tag{17}$$

$$\begin{aligned}
\phi_{[r]}(\pi_1) = ST_{[g-1,2]}(\pi_1) &+ s_{i,2} + t_{i,2} \\
&+ \sum_{d=g+1}^{h-1} (s_{[d,2]} + t_{[d,2]}) + s_{j,2} + t_{j,2} \\
&+ \sum_{d=h+1}^{r} (s_{[d,2]} + t_{[d,2]}) \\
&+ \max\{IT_{[g-1,2]}(\pi_1), \delta_{[g]}(\pi_1), \delta_{[g+1]}(\pi_1), \ldots, \delta_{[r]}(\pi_1)\} \\
&- ST_{[g-1,3]}(\pi_1) - s_{i,3} - t_{i,3} \\
&- \sum_{d=g+1}^{h-1} (s_{[d,3]} + t_{[d,3]}) - s_{j,3} - t_{j,3} \\
&- \sum_{d=h+1}^{r-1} (s_{[d,3]} + t_{[d,3]}) - s_{[r,3]},
\end{aligned} \tag{18}$$

$$\begin{aligned}\phi_{[r]}(\pi_2) = ST_{[g-1,2]}(\pi_2) + s_{j,2} + t_{j,2} + \sum_{d=g+1}^{h-1}(s_{[d,2]} + t_{[d,2]}) + s_{i,2} + t_{i,2} \\ + \sum_{d=h+1}^{r}(s_{[d,2]} + t_{[d,2]}) \\ + \max\{IT_{[g-1,2]}(\pi_2), \delta_{[g]}(\pi_2), \delta_{[g+1]}(\pi_2), \ldots, \delta_{[r]}(\pi_2)\} \\ - ST_{[g-1,3]}(\pi_2) - s_{j,3} - t_{j,3} - \sum_{d=g+1}^{h-1}(s_{[d,3]} + t_{[d,3]}) - s_{i,3} - t_{i,3} \\ - \sum_{d=h+1}^{r-1}(s_{[d,3]} + t_{[d,3]}) - s_{[r,3]}.\end{aligned} \quad (19)$$

It follows from equations (4) and (5) that

$$\delta_{[r]}(\pi_2) \leq \delta_{[r]}(\pi_1) \quad \text{for} \quad r = g \quad (20)$$

since for all realizations of setup and processing times hypothesis (i) $UBs_{j,1}+UBt_{j,1}+UBs_{i,2} \leq LBs_{i,1}+LBt_{i,1}+LBs_{j,2}$ always implies that $s_{j,1}+t_{j,1}+s_{i,2} \leq s_{i,1}+t_{i,1}+s_{j,2}$. In other words, regardless of which values $s_{i,1}, t_{i,1}, s_{j,2}, s_{j,1}, t_{j,1}$,and$s_{i,2}$ take, it is always true that $s_{j,1}+t_{j,1}+s_{i,2} \leq s_{i,1}+t_{i,1}+s_{j,2}$ as long as $UBs_{j,1}+UBt_{j,1}+UBs_{i,2} \leq LBs_{i,1} + LBt_{i,1} + LBs_{j,2}$ is satisfied. Note that before scheduling (i.e., before jobs are processed) we do not know the exact values of $s_{i,1}, t_{i,1}, s_{j,2}, s_{j,1}, t_{j,1}$, and $s_{i,2}$. However, we are assured that $\delta_{[r]}(\pi_2) \leq \delta_{[r]}(\pi_1)$ as long as $UBs_{j,1}+UBt_{j,1}+UBs_{i,2} \leq LBs_{i,1}+LBt_{i,1}+LBs_{j,2}$ holds. This argument is valid for all the cases to be considered throughout the paper. For brevity it will not be repeated.

Furthermore, by equations (8) and (9)

$$\delta_{[r]}(\pi_2) \leq \delta_{[r]}(\pi_1) \quad \text{for} \quad r = g+1, g+2, \ldots, h-1 \quad (21)$$

since hypothesis (i) and (ii) imply that $UBs_{j,1}+UBt_{j,1}+UBs_{i,2}+UBt_{i,2} \leq LBs_{i,1}+LBt_{i,1}+LBs_{j,2}+LBt_{j,2}$, and moreover, $UBs_{j,1}+UBt_{j,1}+UBs_{i,2}+UBt_{i,2} \leq LBs_{i,1}+LBt_{i,1}+LBs_{j,2}+LBt_{j,2}$ implies that $s_{j,1}+t_{j,1}+s_{i,2}+t_{i,2} \leq s_{i,1}+t_{i,1}+s_{j,2}+t_{j,2}$. Also, it follows from equations (12) and (13) that

$$\delta_{[r]}(\pi_2) \leq \delta_{[r]}(\pi_1) \quad \text{for} \quad r = h \quad (22)$$

since $t_{i,2} \leq t_{j,2}$ is always true as long as $UBt_{i,2} \leq LBt_{j,2}$, by hypothesis (ii).

Moreover, for *r= h+1, h+2, ..., n,*

$$\delta_{[r]}(\pi_2) = \delta_{[r]}(\pi_1) \quad \text{for} \quad r = h+1, h+2, \ldots, n \quad (23)$$

by equations (16) and (17). It should be clear that

$$\delta_{[r]}(\pi_2) = \delta_{[r]}(\pi_1) \quad \text{for} \quad r = 1, 2, \ldots, g-1 \quad (24)$$

since both sequences π_1 and π_2 have the same jobs in these positions. Therefore,

$$\delta_{[r]}(\pi_2) \leq \delta_{[r]}(\pi_1) \quad \text{for} \quad r = 1, 2, \ldots, n \quad (25)$$

as a result of equations (20)–(24).

It should be obvious that

$$\phi_{[r]}(\pi_2) = \phi_{[r]}(\pi_1) \quad \text{for} \quad r = 1, 2, \ldots, g-1 \tag{26}$$

as a result of the fact that both sequences have the same jobs in these positions. By hypothesis (iii) $UBs_{j,2} + UBt_{j,2} + UBs_{i,3} \le LBs_{i,2} + LBt_{i,2} + LBs_{j,3}$ which implies that $s_{j,2} + t_{j,2} + s_{i,3} \le s_{i,2} + t_{i,2} + s_{j,3}$, therefore, from equations (6), (7), and (25)

$$\phi_{[r]}(\pi_2) \le \phi_{[r]}(\pi_1) \quad \text{for} \quad r = g. \tag{27}$$

Moreover, it follows from equations (10), (11), and (25) that

$$\phi_{[r]}(\pi_2) \le \phi_{[r]}(\pi_1) \quad \text{for} \quad r = g+1, g+2, \ldots, h-1 \tag{28}$$

since by hypothesis (iii) and (iv) $UBs_{j,2}+UBt_{j,2}+UBs_{i,3} \le LBs_{i,2}+LBt_{i,2}+LBs_{j,3}$ and $UBt_{i,3} \le LBt_{j,3}$ imply $UBs_{j,2} + UBt_{j,2} + UBs_{i,3} + UBt_{i,3} \le LBs_{i,2} + LBt_{i,2} + LBs_{j,3} + LBt_{j,3}$, and $s_{j,2} + t_{j,2} + s_{i,3} + t_{i,3} \le s_{i,2} + t_{i,2} + s_{j,3} + t_{j,3}$ follows from the fact that $UBs_{j,2} + UBt_{j,2} + UBs_{i,3} + UBt_{i,3} \le LBs_{i,2} + LBt_{i,2} + LBs_{j,3} + LBt_{j,3}$. Furthermore, it follows from equations (14), (15), and (25) that

$$\phi_{[r]}(\pi_2) \le \phi_{[r]}(\pi_1) \quad \text{for} \quad r = h \tag{29}$$

since by hypothesis (iv), $UBt_{i,3} \le LBt_{j,3}$ implies that $t_{i,3} \le t_{j,3}$. As a result of equations (18), (19), and (25)

$$\phi_{[r]}(\pi_2) \le \phi_{[r]}(\pi_1) \quad \text{for} \quad r = h+1, h+2, \ldots, n. \tag{30}$$

Therefore, it follows from equations (26)–(30) that

$$\phi_{[r]}(\pi_2) \le \phi_{[r]}(\pi_1) \quad \text{for} \quad r = 1, 2, \ldots, n. \tag{31}$$

It should be noted that for $r = g$

$$C_{[r,3]}(\pi_1) = ST_{[r-1,3]}(\pi_1) + s_{i,3} + t_{i,3} + \max\{IT_{[r-1,3]}(\pi_1), \phi_{[r]}(\pi_1)\},$$
$$C_{[r,3]}(\pi_2) = ST_{[r-1,3]}(\pi_2) + s_{j,3} + t_{j,3} + \max\{IT_{[r-1,3]}(\pi_2), \phi_{[r]}(\pi_2)\}.$$

From the above two equations,

$$C_{[r,3]}(\pi_2) \le C_{[r,3]}(\pi_1) \quad r = g \tag{32}$$

since $s_{j,3} + t_{j,3} \le s_{i,3} + t_{i,3}$ which follows from the fact that $UBs_{j,3} + UBt_{j,3} \le LBs_{i,3}+LBt_{i,3}$ by hypothesis (v). It should be noted that for $r = g+1, g+2, \ldots, h-1$,

$$\begin{aligned} C_{[r,3]}(\pi_1) = {} & ST_{[r-1,3]}(\pi_1) + s_{i,3} + t_{i,3} + \sum_{d=g+1}^{r} (s_{[d,3]} + t_{[d,3]}) \\ & + \max\{IT_{[r-1,3]}(\pi_1), \phi_{[g]}(\pi_1), \phi_{[g+1]}(\pi_1), \ldots, \phi_{[r]}(\pi_1)\}, \\ C_{[r,3]}(\pi_2) = {} & ST_{[r-1,3]}(\pi_2) + s_{j,3} + t_{j,3} + \sum_{d=g+1}^{r} (s_{[d,3]} + t_{[d,3]}) \\ & + \max\{IT_{[r-1,3]}(\pi_2), \phi_{[g]}(\pi_2), \phi_{[g+1]}(\pi_2), \ldots, \phi_{[r]}(\pi_2)\}. \end{aligned}$$

It follows from the last two equations and equation (31) that

$$C_{[r,3]}(\pi_2) \leq C_{[r,3]}(\pi_1) \quad \text{for} \quad r = g+1, g+2, \ldots, h-1 \tag{33}$$

since $s_{j,3} + t_{j,3} \leq s_{i,3} + t_{i,3}$ which follows from the fact that $UBs_{j,3} + UBt_{j,3} \leq LBs_{i,3} + LBt_{i,3}$ by hypothesis (v). Furthermore, for $r = h, h+1, \ldots, n$,

$$\begin{aligned}
C_{[r,3]}(\pi_1) = {} & ST_{[r-1,3]}(\pi_1) + s_{i,3} + t_{i,3} \\
& + \sum_{d=g+1}^{h-1} (s_{[d,3]} + t_{[d,3]}) + s_{j,3} + t_{j,3} \\
& + \sum_{d=h}^{r} s_{[d,3]} + t_{[d,3]}) \\
& + \max \{ IT_{[r-1,3]}(\pi_1), \phi_{[g]}(\pi_1),\ \phi_{[g+1]}(\pi_1), \ldots, \phi_{[h]}(\pi_1), \\
& \qquad \phi_{[h+1]}(\pi_1), \ldots, \phi_{[r]}(\pi_1) \}, \\
C_{[r,3]}(\pi_2) = {} & ST_{[r-1,3]}(\pi_2) + s_{j,3} + t_{j,3} \\
& + \sum_{d=g+1}^{h-1} (s_{[d,3]} + t_{[d,3]}) + s_{i,3} + t_{i,3} \\
& + \sum_{d=h}^{r} (s_{[d,3]} + t_{[d,3]}) \\
& + \max \{ IT_{[r-1,3]}(\pi_2), \phi_{[g]}(\pi_2),\ \phi_{[g+1]}(\pi_2), \ldots, \phi_{[h]}(\pi_2), \\
& \qquad \phi_{[h+1]}(\pi_2), \ldots, \phi_{[r]}(\pi_2) \}.
\end{aligned}$$

Therefore, from the last two equations and equation (31)

$$C_{[r,3]}(\pi_2) \leq C_{[r,3]}(\pi_1) \quad \text{for} \quad r = h, h+1, \ldots, n. \tag{34}$$

It is clear that $C_{[r,3]}(\pi_2) = C_{[r,3]}(\pi_1)$ for $r = 1, 2, \ldots, h-1$ since both jobs have the same jobs in these position. Hence, it follows from this fact and equations (32)–(34) that $TCT(\pi_2) \leq TCT(\pi_1)$. □

Corollary 1. *For the problem* $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} = s_{j,m} = UBs_{j,m}|\sum C_j$*, the sequence* $\Phi_1 = (\phi_1, j, \phi_2, i, \phi_3) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, \phi_2, j, \phi_3) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $s_{j,1} + UBt_{j,1} + s_{i,2} \leq s_{i,1} + LBt_{i,1} + s_{j,2}$,
(ii) $UBt_{i,2} \leq LBt_{j,2}$,
(iii) $s_{j,2} + UBt_{j,2} + s_{i,3} \leq s_{i,2} + LBt_{i,2} + s_{j,3}$,
(iv) $UBt_{i,3} \leq LBt_{j,3}$, *and*
(v) $s_{j,3} + UBt_{j,3} \leq s_{i,3} + LBt_{i,3}$.

Corollary 2. *For the problem* $F3|LBt_{j,m} = t_{j,m} = UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$*, the sequence* $\Phi_1 = (\phi_1, j, \phi_2, i, \phi_3) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, \phi_2, j, \phi_3) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $UBs_{j,1} + t_{j,1} + UBs_{i,2} \leq LBs_{i,1} + t_{i,1} + LBs_{j,2}$,
(ii) $t_{i,2} \leq t_{j,2}$,
(iii) $UBs_{j,2} + t_{j,2} + UBs_{i,3} \leq LBs_{i,2} + t_{i,2} + LBs_{j,3}$,
(iv) $t_{i,3} \leq t_{j,3}$, *and*
(v) $UBs_{j,3} + t_{j,3} \leq LBs_{i,3} + t_{i,3}$.

Corollary 3. *For the problem* $F3|LBt_{j,m} = t_{j,m} = UBt_{j,m}; LBs_{j,m} = s_{j,m} = UBs_{j,m}|\sum C_j$, *the sequence* $\Phi_1 = (\phi_1, j, \phi_2, i, \phi_3) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, \phi_2, j, \phi_3) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $s_{j,1} + t_{j,1} + s_{i,2} \leq s_{i,1} + t_{i,1} + s_{j,2}$,
(ii) $t_{i,2} \leq t_{j,2}$,
(iii) $s_{j,2} + t_{j,2} + s_{i,3} \leq s_{i,2} + t_{i,2} + s_{j,3}$,
(iv) $t_{i,3} \leq t_{j,3}$, *and*
(v) $s_{j,3} + t_{j,3} \leq s_{i,3} + t_{i,3}$.

The proofs of the above three corollaries directly follow from that of Theorem 1.

The dominance relation given in Theorem 1 is called a global dominance relation since when the specified relations hold, say, for jobs i and j (j precedes i), then regardless of the positions of jobs i and j, in an optimal solution job j precedes job i even if they are not adjacent. The relation given in the following theorem is called a local dominance relation. That is because when the specified relations hold, say, for jobs i and j (j precedes i), then job j precedes job i in an optimal solution only when both jobs are adjacent. Notice that a global dominance relation helps in reducing the search space more than a local dominance relation. For a given problem of n jobs, (n-1)! sequences will be eliminated if a single pair satisfies a local dominance relation while n!/2 sequences will be eliminated if a single pair satisfies a global dominance relation. More sequences will be eliminated if more than one pair satisfies the relations.

Theorem 2. *For the problem* $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$, *the sequence* $\Phi_1 = (\phi_1, j, i, \phi_2) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, j, \phi_2) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $UBs_{j,1} + UBt_{j,1} + UBs_{i,2} \leq LBs_{i,1} + LBt_{i,1} + LBs_{j,2}$,
(ii) $UBs_{j,2} + UBt_{j,2} + UBs_{i,3} \leq LBs_{i,2} + LBt_{i,2} + LBs_{j,3}$,
(iii) $UBs_{j,3} + UBt_{j,3} \leq LBs_{i,3} + LBt_{i,3}$, *and*
(iv) *either* $\{UBs_{j,2} + UBt_{j,2} \leq LBs_{j,3} + LBt_{j,3}$
and either $UBs_{i,1} + UBt_{i,1} \leq LBs_{i,2} + LBt_{j,2}$
or $UBs_{j,1} + UBt_{j,1} \leq LBs_{j,2} + LBt_{j,2}\}$
or $\{UBs_{i,2} + UBt_{i,2} \leq LBs_{i,3} + LBt_{j,3}$
and $UBs_{i,1} + UBt_{i,1} \leq LBs_{i,2} + LBt_{j,2}\}$.

Proof. The proof of Theorem 2 is similar to that of Theorem 1 where positions g and h are next to each other. In other words, $h = g + 1$. □

Corollary 4. *For the problem* $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} = s_{j,m} = UBs_{j,m}|\sum C_j$, *the sequence* $\Phi_1 = (\phi_1, j, i, \phi_2) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, j, \phi_2) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $s_{j,1} + UBt_{j,1} + s_{i,2} \le s_{i,1} + LBt_{i,1} + s_{j,2}$,
(ii) $s_{j,2} + UBt_{j,2} + s_{i,3} \le s_{i,2} + LBt_{i,2} + s_{j,3}$,
(iii) $s_{j,3} + UBt_{j,3} \le s_{i,3} + LBt_{i,3}$, *and*
(iv) *either* $\{s_{j,2} + UBt_{j,2} \le s_{j,3} + LBt_{j,3}$ *and either* $s_{i,1} + UBt_{i,1} \le s_{i,2} + LBt_{j,2}$ *or* $s_{j,1} + UBt_{j,1} \le s_{j,2} + LBt_{j,2}\}$ *or* $\{s_{i,2} + UBt_{i,2} \le s_{i,3} + LBt_{j,3}$ *and* $s_{i,1} + UBt_{i,1} \le s_{i,2} + LBt_{j,2}\}$.

Corollary 5. *For the problem* $F3|LBt_{j,m} = t_{j,m} = UBt_{j,m}; LBs_{j,m} \le s_{j,m} \le UBs_{j,m}|\sum C_j$, *the sequence* $\Phi_1 = (\phi_1, j, i, \phi_2) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, j, \phi_2) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $UBs_{j,1} + t_{j,1} + UBs_{i,2} \le LBs_{i,1} + t_{i,1} + LBs_{j,2}$,
(ii) $UBs_{j,2} + t_{j,2} + UBs_{i,3} \le LBs_{i,2} + t_{i,2} + LBs_{j,3}$,
(iii) $UBs_{j,3} + t_{j,3} \le LBs_{i,3} + t_{i,3}$, *and*
(iv) *either* $\{UBs_{j,2} + t_{j,2} \le LBs_{j,3} + t_{j,3}$ *and either* $UBs_{i,1} + t_{i,1} \le LBs_{i,2} + t_{j,2}$ *or* $UBs_{j,1} + t_{j,1} \le LBs_{j,2} + t_{j,2}\}$ *or* $\{UBs_{i,2} + t_{i,2} \le LBs_{i,3} + t_{j,3}$ *and* $UBs_{i,1} + t_{i,1} \le LBs_{i,2} + t_{j,2}\}$.

Corollary 6. *For the problem* $F3|LBt_{j,m} = t_{j,m} = UBt_{j,m}; LBs_{j,m} = s_{j,m} = UBs_{j,m}|\sum C_j$, *the sequence* $\Phi_1 = (\phi_1, j, i, \phi_2) \in \Phi$ *dominates the sequence* $\Phi_2 = (\phi_1, i, j, \phi_2) \in \Phi$ *with respect to PT if the following inequalities hold:*

(i) $s_{j,1} + t_{j,1} + s_{i,2} \le s_{i,1} + t_{i,1} + s_{j,2}$,
(ii) $s_{j,2} + t_{j,2} + s_{i,3} \le s_{i,2} + t_{i,2} + s_{j,3}$,
(iii) $s_{j,3} + t_{j,3} \le s_{i,3} + t_{i,3}$, *and*
(iv) *either* $\{s_{j,2} + t_{j,2} \le s_{j,3} + t_{j,3}$ *and either* $s_{i,1} + t_{i,1} \le s_{i,2} + t_{j,2}$ *or* $s_{j,1} + t_{j,1} \le s_{j,2} + t_{j,2}\}$ *or* $\{s_{i,2} + t_{i,2} \le s_{i,3} + t_{j,3}$ *and* $s_{i,1} + t_{i,1} \le s_{i,2} + t_{j,2}\}$.

The proofs of Corollaries 4-6 directly follow from that of Theorem 2.

5. AN EXAMPLE

Consider a four-job problem given in Table 1 where lower and upper bounds for both setup and processing times of each job are given. Note that these are bounds and we do not know the exact values until the processing of all jobs is completed. We would like to obtain (if possible) the optimal solution or reduce the size of the set which contain the optimal solution by applying the developed dominance relations.

Table 1. Bounds for setup and processing times for the given example

Job k	$LBs_{k,1}$	$LBs_{k,2}$	$LBs_{k,3}$	$LBt_{k,1}$	$LBt_{k,2}$	$LBt_{k,3}$	$UBs_{k,1}$	$UBs_{k,2}$	$UBs_{k,3}$	$UBt_{k,1}$	$UBt_{k,2}$	$UBt_{k,3}$
1	10	15	10	18	6	10	12	16	11	20	9	11
2	1	8	8	1	9	11	3	8	9	2	10	11
3	5	19	18	12	9	8	6	20	18	15	9	8
4	36	29	22	14	6	6	40	36	25	25	8	8

Theorem 1 implies the following precedence relations in an optimal solution for a feasible vector $s \in PT$ of setup times and a feasible vector $t \in PT$ of processing times, polytope PT being defined in Table 1. According to Theorem 1, job 2 precedes jobs 1, 3, and 4 in an optimal solution. This indicates that job 2 has to be in the first position since the only possible way that job 2 precedes jobs 1, 3, and 4 is by placing jobs 1, 3, and 4 in the last three positions indicating that we have a partial sequence as $\Phi = (2, -, -, -)$. Therefore, the optimal solution will be among the remaining 6 sequences: $\Phi_1 = (2, 3, 1, 4)$, $\Phi_2 = (2, 3, 4, 1)$, $\Phi_3 = (2, 4, 1, 3)$, $\Phi_4 = (2, 4, 3, 1)$, $\Phi_5 = (2, 1, 3, 4)$, $\Phi_6 = (2, 1, 4, 3)$. In other words, currently we have the solution set as $\Phi^*\{\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6\}$. Moreover, job 3 precedes job 1 in an optimal solution, which again is as a result of Theorem 1. Therefore, the sequences Φ_3, Φ_5, and Φ_6 can also be eliminated from the set, which results in the solution set as $\Phi^*\{\Phi_1, \Phi_2, \Phi_4\}$. It follows also from Theorem 1 that job 4 should precede job 1 in an optimal solution. Therefore, the sequence Φ_1 can also be eliminated from the set resulting in the solution set as $\Phi^*\{\Phi_2, \Phi_4\}$ for the problem $F3|LBt_{j,m} \leq t_{j,m} \leq UBt_{j,m}; LBs_{j,m} \leq s_{j,m} \leq UBs_{j,m}|\sum C_j$ with the data given in Table 1.

Table 2. Four possible realizations for the given example

Realization	Job k	$s_{k,1}$	$s_{k,2}$	$s_{k,3}$	$t_{k,1}$	$t_{k,2}$	$t_{k,3}$
	1	10	15	10	18	6	10
1	2	1	8	8	1	9	11
	3	5	19	18	12	9	8
	4	36	29	22	14	6	6
	1	12	16	11	20	9	11
2	2	3	8	9	2	10	11
	3	6	20	18	15	9	8
	4	40	36	25	25	8	8
	1	11	15	11	19	7	10
3	2	2	8	8	2	9	11
	3	6	19	18	13	9	8
	4	38	33	23	18	7	7
	1	12	16	10	18	8	11
4	2	1	8	9	2	10	11
	3	5	20	18	14	9	8
	4	39	31	24	20	8	6

Table 3. The optimal solution for the realizations given in Table 2

Realization	$TCT(\Phi_2)$	$TCT(\Phi_4)$
1	284	303
2	334	373
3	301	332
4	309	346

Depending on the realization of setup and processing times, either the sequence Φ_2, or Φ_4, or both will be optimal. Four different realizations of the setup and processing times of the problem described in Table 1 are given in Table 2.

The objective function values of both sequences for all the four realizations described in Table 2 are given in Table 3.

It is clear that the sequence Φ_2 is optimal for all the four realizations. It should be noted that for some realizations the sequence Φ_4 might be optimal while for some other realizations both Φ_2 and Φ_4 might be optimal.

6. COMPUTATIONAL EXPERIMENTS

In this section, the effectiveness of the developed dominance relations from Theorem 1 (GDR) and Theorem 2 (LDR) is investigated on randomly generated problem instances. The upper bounds for processing and setup times were randomly generated from uniform distributions with $UBt_{i,j}$ from $[1, 100]$ and $UBs_{i,j}$ from $[0, 100k]$. The parameter k is the expected ratio of setup time to processing time $(s_{i,j}/t_{i,j})$. The k value for each data set was set to 0.5, 1.0, and 1.5. The lower bounds $LBt_{i,j}$ on processing times were generated from $LBt_{i,j} = UBt_{i,j} - \Delta_t$ where Δ_t values were randomly generated from uniform distribution from three different ranges, namely, $U(1,5)$, $U(1,10)$, and $U(1,15)$. Similarly, the lower bounds $LBs_{i,j}$on setup times were generated from $LBs_{i,j} = UBs_{i,j} - \Delta_s$ where Δ_s values were randomly generated from uniform distribution from three different ranges, namely, $U(0,5k)$, $U(0,10k)$, and $U(0,15k)$. It should be noted that in the cases where $LBt_{i,j}(LBs_{i,j})$ was less than 1(0), $LBt_{i,j}(LBs_{i,j})$ was assigned a value of 1(0).

The following algorithm is used to find the number of LDR and GDR relationships.

Step 1: Suppose the sequence of jobs is given by π=(1, 2,. . . ,n)

Step 2: Set $k = 1$, $t = 2$, N_{LDR}=N_{GDR}=0

Step 3: Select the jobs in positions k and t of π
If the jobs in positions k and t satisfy the conditions of Theorem 1,
then $N_{GDR} = N_{GDR} + 1$
If the jobs in positions k and t satisfy the conditions of Theorem 2,
then $N_{LDR} = N_{LDR} + 1$
Let $t = t + 1$
If $t \leq n$,
then Go to Step 3

Step 4: Let $k = k + 1$ and $t = k + 1$
If $k < n$, Go to Step 3

Step 5: Stop. N_{GDR} and N_{LDR} denote the number of GDR and LDR relationships, respectively.

Since we have two nested loops that are both proportional to n, the overall computational complexity of this algorithm can be shown to be $O(n^2)$. The total number of relationships is $n(n-1)/2$.

Thirty replicates were generated for each instance. The results are summarized in Table 4. In the table, the first column (n) denotes the number of jobs, the second (Δ_t) and third (Δ_s) columns denote the difference between the upper and lower bounds of processing and setup times, respectively. The next two columns represent the average number of relationships satisfying the conditions in Theorem 1 (GDR) and in Theorem 2 (LDR) out of thirty replicates for $k = 0.5$. The next two columns give the average GDR and LDR values for the case when $k = 1$ and the final two columns show the average GDR and LDR values for $k = 1.5$.

It should be noted that for a given problem of n jobs, $(n-1)!$ sequences will be eliminated (from the dominating schedules set) if a single pair satisfies a local dominance relation (Theorem 2) while $n!/2$ sequences will be eliminated if a single pair satisfies a global dominance relation (Theorem 1). More sequences will be eliminated if more than one pair satisfies the relations. Therefore, as can bee seen from the average values for LDR and GDR (Tab. 4), both LDR and GDR help reduce the size of dominating schedules set.

Table 4. Computational Results

			k=0.5		k=1		k=1.5	
n	Δ_t	Δ_s	LDR	GDR	LDR	GDR	LDR	GDR
40	U(1,5)	U(0,5k)	25.95	24.51	25.90	24.66	25.91	24.86
40	U(1,5)	U(0,10k)	25.85	24.54	25.77	24.56	25.86	24.77
40	U(1,5)	U(0,15k)	25.84	24.67	25.80	24.34	25.86	24.41
40	U(1,10)	U(0,5k)	25.89	23.30	25.86	22.41	25.78	23.71
40	U(1,10)	U(0,10k)	25.83	23.15	25.76	23.02	25.69	22.69
40	U(1,10)	U(0,15k)	25.66	22.70	25.78	23.24	25.71	23.64
40	U(1,15)	U(0,5k)	25.73	21.65	25.69	22.57	25.82	22.00
40	U(1,15)	U(0,10k)	25.48	21.59	25.55	21.66	25.53	21.38
40	U(1,15)	U(0,15k)	25.41	22.13	25.24	21.53	25.67	22.09
60	U(1,5)	U(0,5k)	58.79	55.94	58.86	56.19	58.73	55.25
60	U(1,5)	U(0,10k)	58.59	55.09	58.86	56.00	58.53	55.80
60	U(1,5)	U(0,15k)	58.42	55.86	58.70	55.91	58.96	56.14
60	U(1,10)	U(0,5k)	58.71	53.08	58.56	52.26	58.66	52.39
60	U(1,10)	U(0,10k)	58.50	52.37	58.20	52.16	58.44	52.76
60	U(1,10)	U(0,15k)	57.71	51.56	58.09	52.01	58.20	51.98
60	U(1,15)	U(0,5k)	58.06	49.72	58.66	50.46	58.66	50.19
60	U(1,15)	U(0,10k)	57.94	49.83	58.04	49.77	58.10	49.45
60	U(1,15)	U(0,15k)	57.89	49.42	57.99	48.36	58.01	49.40
80	U(1,5)	U(0,5k)	104.84	99.47	105.13	99.63	105.05	100.53
80	U(1,5)	U(0,10k)	104.67	99.90	104.96	99.51	104.89	99.94
80	U(1,5)	U(0,15k)	104.23	98.64	104.83	99.43	104.40	99.25
80	U(1,10)	U(0,5k)	104.40	94.24	104.68	93.41	104.87	94.95
80	U(1,10)	U(0,10k)	103.83	91.92	104.31	92.58	104.59	94.97
80	U(1,10)	U(0,15k)	104.14	94.63	104.31	93.09	104.37	94.56
80	U(1,15)	U(0,5k)	103.61	88.87	104.40	89.86	104.60	88.23
80	U(1,15)	U(0,10k)	103.52	88.33	104.33	89.17	104.18	89.27

Table 4 (continued)

			k=0.5		k=1		k=1.5	
n	Δ_t	Δ_s	LDR	GDR	LDR	GDR	LDR	GDR
80	U(1,15)	U(0,15k)	102.90	86.72	103.63	87.62	102.74	88.41
100	U(1,5)	U(0,5k)	164.18	155.79	164.63	155.84	164.88	157.07
100	U(1,5)	U(0,10k)	163.69	155.62	164.15	154.60	163.94	155.53
100	U(1,5)	U(0,15k)	163.96	156.17	163.82	155.17	164.24	155.12
100	U(1,10)	U(0,5k)	163.67	145.92	164.19	146.61	163.64	148.36
100	U(1,10)	U(0,10k)	162.63	145.39	163.24	147.60	163.46	146.72
100	U(1,10)	U(0,15k)	162.37	146.50	163.09	146.24	162.75	145.45
100	U(1,15)	U(0,5k)	163.13	136.47	163.72	138.39	164.09	141.01
100	U(1,15)	U(0,10k)	162.31	137.24	162.93	138.59	162.92	139.22
100	U(1,15)	U(0,15k)	162.15	138.48	162.61	136.28	162.71	137.17
120	U(1,5)	U(0,5k)	237.56	226.68	236.93	226.41	237.46	225.03
120	U(1,5)	U(0,10k)	235.84	222.99	236.08	225.31	237.02	223.94
120	U(1,5)	U(0,15k)	236.09	221.40	236.06	226.32	236.70	222.78
120	U(1,10)	U(0,5k)	236.17	211.35	236.59	213.55	236.24	214.19
120	U(1,10)	U(0,10k)	235.93	211.45	235.61	209.67	234.64	213.04
120	U(1,10)	U(0,15k)	234.39	210.06	233.96	208.48	235.65	211.91
120	U(1,15)	U(0,5k)	234.69	202.63	235.89	200.54	236.45	202.18
120	U(1,15)	U(0,10k)	234.13	197.49	234.33	198.47	235.03	197.16
120	U(1,15)	U(0,15k)	231.94	199.63	234.49	198.71	233.82	199.23

The GDR and LDR relationships seem to be not affected by the parameter k (the ratio of setup times to processing times). As such the GDR and LDR relationships are not also affected by Δ_s. On the other hand, both relationships are slightly affected by Δ_t showing that as the difference between the lower and upper bound of processing times increases, the average number of LDR and GDR slightly decreases. This is expected since as the difference between the two values increases it becomes harder to satisfy the conditions of LDR and GDR.

7. CONCLUSIONS

In this paper, we addressed the three machine flowshop scheduling problem to minimize total completion time where setup times are treated as separate from processing times as apposed to assuming that they are included in processing times. Moreover, we modeled both setup and processing times as random variables rather than deterministic and fixed values where only lower and upper bounds are known for both setup and processing times of each job. For such flowshop scheduling problems, there may not exist a unique schedule that remains optimal for all possible realizations of setup and processing times. Therefore, a global dominance relation and a local dominance relation which help reduce the size of dominating schedules have been developed in this paper. The use of the developed dominance relations was illustrated by a numerical example.

The developed dominance relations were evaluated by computational experiments on randomly generated problem instances. The computational experiments indicate that the developed dominance relations help in reducing the dominating schedules set.

One possible extension to the problem studied in this paper is to consider the problem with respect to other objective functions such as maximum lateness, or job waiting time variance, e.g., Li et al., 2007. In this paper, it was assumed that setup times are sequence-independent. This assumption is valid for many scheduling environments. However, the assumption may not be valid for some other scheduling environments, e.g., Chandrasekaran et al. (2007).

Therefore, another possible extension is to consider the problem addressed in this paper with sequence-dependent random and bounded setup times.

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