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A Reference Point Approach to Bi-Objective Dynamic Portfolio Optimization

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Abstract. The portfolio selection problem presented in this paper is formulated as a bi-objective mixed integer program. The portfolio selection problem considered is based on a dynamic model of investment, in which the investor buys and sells securities in successive investment periods. The problem objective is to dynamically allocate the wealth on different securities to optimize by reference point method the portfolio expected return and the probability that the return is not less than a required level. In computational experiments the dataset of daily quotations from the Warsaw Stock Exchange were used.

Keywords: Dynamic Portfolio, Mixed Integer Programming, Reference Point Method, Bi-Objective Optimization, Value-at-Risk

Mathematics Subject Classification: 90C11, 90C90, 91G10

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1. INTRODUCTION

The optimal security selection is a classical portfolio problem since the seminal work of Markowitz (Markowitz, 1952, 1997). In the standard approach, the decision maker selects the securities in such a way that the portfolio expectation is maximized, under the constraint that risk (variance) must be kept under a fixed threshold (Benati, 2007, Lin, 2009). The problem consists in picking the best amount of securities, with the aim of maximizing future returns. It is a typical multivariate problem: the only way to improve future returns is to increase the risk level that the decision maker is disposed to accept (Ogryczak, 2000, Young, 1998).

In Markowitz's approach the problem is formulated as an optimization problem involving two criteria: the reward of a portfolio, which is measured by the mean and should be maximized, and the risk of the portfolio (measured by the variance of return) that should by minimized. In the presence of two criteria there is not a single optimal solution (portfolio), which represents the tradeoff between risk and return (Anagnostopoulos, 2010).

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While the original Markowitz model forms a quadratic programming problem, many attempts have been made to linearize the portfolio optimization procedure (Sawik, 2009a, 2009b, 2009c, 2009d, 2009e, 2009f, 2009g, 2008, Speranza, 1993, Young, 1998). The linear program solvability is very important for applications to real-life financial and other decisions where the constructed portfolios have to meet numerous side constraints. Examples of them are minimum transaction lots, transaction costs or mutual funds characteristics etc. The introduction of these features leads to mixed integer program problems. This paper presents a bi-criterion extension of the Markowitz portfolio optimization model, in which the variance has been replaced with the Value-at-Risk (VaR). The VaR is a quantile of the return distribution function (Benati, 2007, Sawik, 2009a, 2008).

The advantage of using VaR measure in portfolio optimization is that this value of risk is independent of any distribution hypothesis. It concerns only downside risk, namely the risk of loss. This index measures the loss in question in a certain way. Finally VaR is valid for all types of securities and therefore either involve the various valuation models or be independent of these models (Esch, 2005, Gaivoronski 2005).

This portfolio optimization problem is formulated as a bi-objective mixed integer program. The portfolio selection problem considered is based on a dynamic model of investment, in which the investor buys and sells securities in successive investment periods. The problem objective is to dynamically allocate the wealth on different securities to optimize by reference point method of the portfolio expected return and the probability that the return is not less than a required level.

The results of some computational experiments with the mixed integer programming approach modeled on a real data from the Warsaw Stock Exchange are reported. The input dataset consist of time series of the daily quotation of returns of securities from the Warsaw Stock Exchange.

2. REFERENCE POINT METHOD

The reference point method is based on the Tchebycheff metric (Alves, 2007, Bowman, 1976). Let us denote by $\left\|f(x)-\underline{f}\right\|_\lambda$ the λ-weighted Tchebycheff metric, i.e., $\min_{1\leqslant l\leqslant q}\left\{\lambda_l\left|f_l(x)-\underline{f}\right|\right\}$, where $\lambda_l \geqslant 0 \quad \forall \sum_{l=1}^{q}\lambda_l = 1$, and $\underline{f}$ denotes a reference point of the criteria space. Considering $f(x) > \underline{f}$ for all $x \in X$, it has been proven (Bowman, 1976) that the parametrization on λ of $\min_{x\in X}\left\|f_l(x)-\underline{f}\right\|_\lambda$ generates the non-dominated set. The program $\min_{x\in X}\left\|f_l(x)-\underline{f}\right\|_\lambda$ may yield weakly non-dominated solutions, which can be avoided by considering the *augmented weighted Tchebycheff* program:

$$\begin{aligned}
\text{Minimize} \quad & \delta + \gamma\sum_{l=1}^{q} f_l(x) \\
\text{subject to} \quad & \lambda_l\left(f_l(x)-\underline{f}\right) \leqslant \delta, \qquad 1 \leqslant l \leqslant q \\
& x \in X, \\
& \delta \geqslant 0,
\end{aligned}$$

where γ is a small positive value. It has been proven (Steuer, 1986) that there always exists γ small enough that enable to reach all the non-dominated set for the finite-discrete and polyhedral feasible region cases (Alves, 2007).

3. PROBLEM FORMULATION

Let n be the number of securities available in the market with historical quotations in t investment periods, each consisting of h historical periods.

Let r_{ij} be the random variable representing the future daily return of security j in historical time period i.

The portfolio optimization problem with Value-at-Risk constraint is formulated as the classic Markowitz approach, but with Value-at-Risk instead of variance as a risk measure.

Table 1. *Notation*

		Indices
i	=	historical time period i, $i = 1, \dots, m$ (i.e. day)
j	=	security j, $j = 1, \dots, n$
k	=	historical successive investment period k, $k = 1, \dots, t$ (i.e. year, quarter or month, etc)
		Input parameters
h	=	number of historical quotations in each successive investment period
p_i	=	probability assigned to the occurrence of past realization i
r_{ij}	=	observed return of security j in historical time period i
r^{Min}	=	minimum return observed in the market
r^{VaR}	=	return Value-at-Risk
v	=	accepted number of securities in portfolio in each successive investment period
λ	=	weight for the objective functions f_1
γ	=	small positive value
f_1^{opt}	=	ideal solution value of average return
f_2^{opt}	=	ideal solution value of average risk probability
		Variables
x_{jk}	=	percentage of capital invested in successive investment period k in security j
x_{jk}^{buy}	=	percentage of capital invested in successive investment period k for bought security j
x_{jk}^{sell}	=	percentage of capital invested in successive investment period k for sold security j
	=	1, if return of portfolio in historical time period i of successive investment period k is not less than r^{VaR} 0, otherwise
α_k^{VaR}	=	probability that return of investment is not less than r^{VaR} in successive investment period k
z_{jk}	=	1, if in successive investment period k capital is invested in stock j 0, otherwise
δ	=	deviation from the reference solution

The decision maker fixes the lower bound r^{VaR} for successful returns – any investments whose Value-at-Risk is less than r^{VaR} will be not acceptable.

Let r^{Min} be the minimum return that can be observed in the market, for example the biggest possible loss of money invested in portfolio. In the worst case it is the whole amount of capital, so for instance it can be equal – 100% (Benati, 2007, Lin, 2009).

The bi-objective dynamic portfolio optimization model with Value-at-Risk is an NP-hard problem even when future returns are described by discrete uniform distributions (Ehrgott, 2000, Nemhauser, 1999, Steuer, 1986).

The seven types of variables for each successive investment period are introduced in the model: a continuous wealth allocation variable that represents the percentage of wealth allocated to each security, a continuous wealth allocation variable for buying amount of each security, a continuous wealth allocation variable for selling amount of each security, a binary selection variable that prevents the choice of portfolios whose *VaR* is below the fixed threshold and a binary selection variable for selecting each security to the portfolio and finally continuous variable represents deviation from the reference solution.

4. OPTIMIZATION MODEL

In the approach proposed in this paper, the portfolio optimization problem is formulated as reference point method dynamic bi-objective mixed integer program, which allows commercially available software (e.g. AMPL/CPLEX (Fourer, 1990) to be applied for solving medium size, yet practical instances. The problem formulation is presented below.

Minimize

$$\delta + \gamma \left(-\sum_{k=1}^{t} \left(\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk} \right) + \sum_{k=1}^{t} \alpha_k^{aVaR} \right) \tag{1}$$

subject to

$$\lambda \left(-\sum_{k=1}^{t} \left(\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk} \right) + f_1^{opt} \right) \leqslant \delta \tag{2}$$

$$(1-\lambda) \left(\sum_{k=1}^{t} \alpha_k^{aVaR} - f_2^{opt} \right) \leqslant \delta \tag{3}$$

$$y_{ik} \leqslant \frac{\sum\limits_{j=1}^{n} r_{ij} x_{jk} - r^{Min}}{r^{VaR} - r^{Min}}, \qquad i = (k-1)h+1, \ldots, kh, \quad k = 1, \ldots, t \tag{4}$$

$$y_{ik} \geqslant \frac{\sum\limits_{j=1}^{n} r_{ij} x_{jk} - r^{Min}}{r^{VaR} - r^{Min}} - 1, \quad i = (k-1)h+1, ..., kh, \quad k = 1, \ldots, t \tag{5}$$

$$\sum_{i=(k-1)h+1}^{kh} p_i \left(1 - y_{ik}\right) \leqslant \alpha_k^{\alpha VaR}, \quad k = 1, \ldots, t \tag{6}$$

The objective function (1) represents the weighted deviation from the reference point for the portfolio expected return and the probability that the return is not less than a required level. The deviations are defined in constraints (2), (3).

Constraints (4), (5) and (6) prevent the choice of portfolios whose VaR is below the fixed threshold. Every time expected portfolio return is below r^{VaR}, then y_{ik} must be equal to 0 and $1 - y_{ik} = 1$ in constraint (6). Therefore, all probabilities of events i whose returns are below the VaR threshold was summed up. If the result is greater than α^{VaR}, then the portfolio is not feasible.

$$\sum_{j=1}^{n} x_{jk} = 1, \quad k = 1, \ldots, t \tag{7}$$

Constraint (7) requires that in each investment period all capital must be allocated on different securities with positive expected return.

$$x_{j1}^{buy} = x_{j1}, \qquad j = 1, \ldots, n : \sum_{i=1}^{h} p_i r_{ij} > 0 \tag{8}$$

$$x_{j1}^{sell} = 0, \qquad j = 1, \ldots, n \tag{9}$$

$$x_{jk} = x_{jk-1} + x_{jk}^{buy} - x_{jk}^{sell}, \qquad j = 1, \ldots, n, \quad k = 2, \ldots, t \tag{10}$$

Constraints (8), (9) and (10) are responsible for a dynamic balance among x_{jk}, x_{jk}^{buy}, x_{jk}^{sell} for each successive investment period k.

$$\sum_{j=1}^{n} z_{jk} \geqslant v, \quad k = 1, ..., t \tag{11}$$

Constraint (11) ensures that the number of stocks in optimal portfolio must be greater than or equal to the accepted number of assets in the selected portfolio.

$$\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk} >= r^{VaR}, \quad k = 1, ..., t \tag{12}$$

Constraint (12) imposes the minimum portfolio expected return equal r^{VaR} that the decision maker is prepared to accept for each successive investment period k.

$$x_{jk} \leqslant z_{jk}, \quad j = 1, ..., n : \sum_{i=(k-1)h+1}^{kh} p_i r_{ij} > 0, \; k = 1, ..., t \tag{13}$$

$$x_{jk}^{buy} \leqslant z_{jk}, \quad j = 1, ..., n : \sum_{i=(k-1)h+1}^{kh} p_i r_{ij} > 0, \; k = 1, ..., t \tag{14}$$

$$x_{jk}^{sell} \leqslant z_{jk}, \quad j = 1, ..., n, \; k = 1, ..., t \tag{15}$$

Constraints (13), (14) and (15) are responsible for relations between variables x_{jk}, x_{jk}^{buy}, x_{jk}^{sell} and z_{jk} .

$$\frac{z_{jk}}{100} \leqslant x_{jk}, \quad j = 1, ..., n : \sum_{i=(k-1)h+1}^{kh} p_i r_{ij} > 0, \; k = 1, ..., t \tag{16}$$

$$\frac{z_{jk}}{100} \leqslant x_{jk}^{buy}, \quad j = 1, ..., n : \sum_{i=(k-1)h+1}^{kh} p_i r_{ij} > 0, \; k = 1, ..., t \tag{17}$$

Constraints (16) and (17) ensure the addition to portfolio and buying of some amount of security j in successive investment period k.

$$0 \leqslant \alpha_k^{VaR} \leqslant 1, \quad k = 1, ..., t \tag{18}$$

Constraint (18) defines continuous variable α_k^{VaR} – probability that return of investment is not less than r^{VaR} of successive investment period k.

$$x_{jk} = 0, \quad j = 1, ..., n : \sum_{i=(k-1)h+1}^{kh} p_i r_{ij} \leqslant 0, \; k = 1, ..., t \tag{19}$$

Constraint (19) defines continuous variable x_{jk} – percentage of capital invested in successive investment period k in security j and, in addition, eliminates securities with a non-positive expected return.

$$x_{jk}^{buy} = 0, \quad j = 1, ..., n : \sum_{i=(k-1)h+1}^{kh} p_i r_{ij} \leqslant 0, \; k = 1, ..., t \tag{20}$$

$$x_{jk}^{sell} \geqslant 0, \quad j = 1, ..., n, \; k = 1, ..., t \tag{21}$$

$$y_{ik} \in \{0, 1\}, \quad i = (k-1)h + 1, ..., kh, \; k = 1, ..., t \tag{22}$$

$$z_{jk} \in \{0, 1\}, \quad j = 1, ..., n, \; k = 1, ..., t \tag{23}$$

$$\delta \geqslant 0 \tag{24}$$

Finally, constraints (20)–(24) define variables x_{jk}^{buy}, x_{jk}^{sell}, z_{jk}, δ and eliminate securities with a non-positive expected return. Variables x_{jk} are percentage of capital invested in security j in successive investment period k.

The combination of continuous variables x_{jk}, x_{jk}^{buy}, x_{jk}^{sell} and binary variables y_{ik}, z_{jk} leads an NP-hard mixed integer programming problem (Nemhauser, 1999). If the number of historical observations m is bounded by a constant, there are 2^m ways of fixing the variables y_{ik}, z_{jk} for each successive investment period k.

5. COMPUTATIONAL RESULTS

In this section numerical examples and some computational results are presented to illustrate possible applications of the proposed formulations of this multi-period optimization model. The examples are modeled on a real data form the Warsaw Stock Exchange.

Suppose that n securities with historical quotations in t investment periods, each of h days, in total 4020 samples. The eighteen years horizon from 30th Jan 1991 to 30th Jan 2009 – consists of $m = 4020$ historic daily quotations divided into t=20 investment periods ($h = 201$ daily quotations each), with the selection of $n = 241$ input securities for portfolio, quoted each day in the historical horizon. Probability of realization for expected securities returns is the same for each day and summed up for whole period to one. The accepted number of securities in portfolio is at least one security in each successive investment period. The basic parameters for the reference point method take on the following values: $f_1^{opt} = 1$, $f_2^{opt} = 0.05$, $\lambda = 0.5$, $\gamma = 0.01$.

The computational experiments have been performed using AMPL programming language (Fourer, 1990) and the CPLEX v.11 solver (with the default settings) on a laptop with Intel® Core 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM. The computational time limit was set to 10800 CPU seconds (three CPU hours).

Table 2 presents the solution results for objective function α_k^{VaR} probability that return of investment is not less than in each successive investment period k. Table 3 presents the solution results for the objective function of expected portfolios return in each successive investment period k.

Table 4 presents number of securities in the computed portfolios for each successive investment period k.

Table 5 presents computational time range and the solution values of δ – the deviation from the reference solution. In the table 5, column "MIP simplex iteration" shows the number of mixed integer programming simplex iterations until presented solution. Column "B–&–B" shows the number of searched nodes in the branch and bound tree until presented solution. Column "CPU" shows CPU seconds required for proving optimality on a laptop with Intel® Core 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM using the solver CPLEX v.11.

Table 2. *The solution results for objective function* α_k^{VaR}

r^{VaR}	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
historical successive investment period k	Computational results for objective function α_k^{VaR} – probability that return of investment is not less than r^{VaR} in successive investment period k													
1	0.000249	0.008955	0.011940	0.014179	0.015920	0.015920	0.016169	0.016915	0.017413	0.017413	0.017413	0.017413	0.017413	0.017413
2	0.000498	0.007960	0.010199	0.011940	0.012687	0.013433	0.014428	0.014925	0.015920	0.016169	0.016418	0.017413	0.017910	0.017910
3	0.000000	0.000995	0.002736	0.007214	0.008458	0.008706	0.008955	0.009204	0.010199	0.011443	0.012189	0.022886	0.021144	0.023632
4	0.000000	0.000000	0.000000	0.001741	0.003483	0.004229	0.005721	0.008458	0.008955	0.008458	0.011940	0.013433	0.015423	0.021891
5	0.000000	0.002736	0.003731	0.009453	0.010697	0.012189	0.013682	0.015174	0.015423	0.011194	0.012935	0.019154	0.020149	0.016667
6	0.000000	0.000249	0.000249	0.002239	0.002488	0.004478	0.006965	0.006716	0.007463	0.010697	0.013682	0.017910	0.020149	0.024627
7	0.000249	0.001244	0.002239	0.004726	0.005473	0.007214	0.007463	0.010199	0.013184	0.015920	0.019403	0.025373	0.022139	0.022637
8	0.000000	0.000000	0.000000	0.003234	0.003483	0.003483	0.004726	0.003980	0.006468	0.006965	0.009204	0.017910	0.014925	0.023881
9	0.000000	0.000249	0.000746	0.008955	0.011940	0.012438	0.013930	0.012935	0.015423	0.015672	0.016667	0.017910	0.013930	0.022637
10	0.000000	0.000000	0.000000	0.001493	0.002985	0.004975	0.004229	0.005721	0.005473	0.009701	0.010199	0.013930	0.012687	0.017164
11	0.000000	0.000249	0.000746	0.004229	0.002736	0.006716	0.002985	0.001990	0.009701	0.010448	0.012189	0.015174	0.016667	0.021642
12	0.000000	0.000249	0.000000	0.000746	0.001990	0.001741	0.000995	0.001990	0.004229	0.006468	0.009453	0.008209	0.015920	0.021642
13	0.000498	0.000746	0.001493	0.002736	0.003980	0.004975	0.005721	0.006468	0.007711	0.008706	0.010945	0.014179	0.011443	0.018657
14	0.000000	0.000000	0.000249	0.004478	0.002239	0.005473	0.005970	0.009950	0.008706	0.009950	0.012687	0.008209	0.013184	0.021891
15	0.000498	0.002239	0.004229	0.007463	0.009701	0.009950	0.011940	0.011194	0.012935	0.014179	0.017164	0.008706	0.020149	0.012438
16	0.000000	0.000995	0.002736	0.009701	0.010697	0.011194	0.013682	0.014677	0.014925	0.016169	0.018159	0.012687	0.021891	0.019403
17	0.000249	0.000498	0.002488	0.008458	0.009701	0.009453	0.011692	0.014428	0.015672	0.016667	0.017413	0.007214	0.008955	0.021393
18	0.000000	0.000000	0.000995	0.002985	0.003483	0.003731	0.005473	0.006965	0.006965	0.008706	0.009950	0.005473	0.007214	0.025373
19	0.000249	0.000249	0.000995	0.002239	0.002239	0.004726	0.005970	0.006219	0.007214	0.007711	0.011692	0.014179	0.022139	0.022886
20	0.000000	0.000000	0.000498	0.002488	0.002736	0.003980	0.004726	0.006468	0.007960	0.009453	0.011940	0.014677	0.016667	0.018159

Average value

-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
0.00012	0.00138	0.00231	0.00553	0.00636	0.00745	0.00827	0.00923	0.01060	0.01160	0.01358	0.01460	0.01650	0.02060

Maximal value

0.00050	0.00896	0.01194	0.01418	0.01592	0.01592	0.01617	0.01692	0.01741	0.01741	0.01940	0.02537	0.02214	0.02537

Minimal value

0.00000	0.00000	0.00000	0.00075	0.00199	0.00174	0.00100	0.00199	0.00423	0.00647	0.00920	0.00547	0.00721	0.01244

Table 3. *The solution results for objective function – the expected portfolios return*

r^{VaR}	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
historical successive investment period k	Computational results for objective function $\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk}$ – expected portfolios return													
1	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724
2	0.070761	0.070761	0.070761	0.067763	0.067635	0.067371	0.067544	0.070448	0.069815	0.067240	0.067694	0.030125	0.028664	0.028664
3	0.024444	0.024444	0.024444	0.023322	0.024444	0.024444	0.024034	0.024444	0.023868	0.022673	0.023463	0.002359	0.005598	0.002089
4	0.027760	0.026885	0.025954	0.022859	0.023592	0.022952	0.023908	0.026289	0.025925	0.019273	0.022895	0.007076	0.017101	0.010134
5	0.055988	0.055465	0.055966	0.055219	0.053645	0.054111	0.053835	0.055465	0.053127	0.040806	0.040221	0.023383	0.011053	0.022263
6	0.016764	0.015921	0.014679	0.012419	0.010556	0.011936	0.013085	0.012418	0.010993	0.012238	0.011124	0.004557	0.000996	0.001355
7	0.020589	0.020589	0.019941	0.016857	0.016613	0.017639	0.015481	0.015616	0.016691	0.015112	0.002549	0.012430	0.011628	0.003331
8	0.031882	0.031354	0.030397	0.029042	0.027955	0.027076	0.026588	0.025508	0.026435	0.026012	0.025394	0.011218	0.019939	0.006816
9	0.048203	0.043892	0.039017	0.043258	0.045281	0.044065	0.044344	0.040406	0.045859	0.044854	0.045469	0.048014	0.013503	0.006938
10	0.028001	0.027716	0.026827	0.024002	0.025275	0.025197	0.022398	0.022944	0.022575	0.022790	0.018884	0.007076	0.003166	0.008150
11	0.025639	0.026031	0.026184	0.024804	0.020658	0.024065	0.018287	0.016384	0.022287	0.021258	0.021400	0.011683	0.000835	0.005386
12	0.025682	0.025682	0.025548	0.024555	0.024519	0.024356	0.021084	0.021295	0.023018	0.019324	0.020103	0.002275	0.005586	0.004849
13	0.056018	0.056018	0.055462	0.053231	0.052851	0.052570	0.052162	0.051774	0.050749	0.046546	0.047800	0.028891	0.017758	0.008715
14	0.058699	0.056988	0.055645	0.055334	0.052905	0.053225	0.053446	0.056469	0.052085	0.051592	0.051437	0.036650	0.025911	0.023462
15	0.071572	0.073183	0.073669	0.075568	0.076221	0.076269	0.076789	0.073240	0.076417	0.073869	0.076417	0.008486	0.008165	0.009565
16	0.080490	0.081758	0.080721	0.080926	0.078377	0.077874	0.080885	0.080420	0.078805	0.075332	0.077503	0.009849	0.024172	0.015486
17	0.085573	0.085815	0.085058	0.084567	0.082168	0.079443	0.082134	0.083629	0.082977	0.081912	0.080363	0.049416	0.039606	0.008291
18	0.076582	0.075889	0.075833	0.075581	0.073863	0.072654	0.074755	0.073789	0.072593	0.069594	0.069667	0.049453	0.050654	0.050313
19	0.024858	0.024246	0.023692	0.018291	0.015745	0.017869	0.018093	0.016482	0.015297	0.012908	0.014121	0.010479	0.007908	0.024858
20	0.017350	0.016878	0.017081	0.015555	0.013707	0.014389	0.012835	0.013079	0.013483	0.009589	0.010298	0.014722	0.001857	0.003602

Expected portfolio return for whole investment period $\sum_{k=1}^{t}\left(\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk}\right)$

-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
0.86458	0.85724	0.84460	0.82088	0.80374	0.80523	0.79941	0.79782	0.80072	0.75064	0.74452	0.38587	0.31182	0.26199
Average value of expected portfolio return													
0.04323	0.04286	0.04223	0.04104	0.04019	0.04026	0.03997	0.03989	0.04004	0.03753	0.03723	0.01929	0.01559	0.01310
Maximal value of expected portfolio return													
0.08557	0.08581	0.08506	0.08457	0.08217	0.07944	0.08213	0.08363	0.08298	0.08191	0.08036	0.04945	0.05065	0.05031
Minimal value of expected portfolio return													
0.01676	0.01592	0.01468	0.01242	0.01056	0.01194	0.01283	0.01242	0.01099	0.00959	0.00255	0.00227	0.00083	0.00135

Table 4. *The solution results for objective function – the expected portfolios return*

r^{VaR}	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
historical successive investment period k	Number of securities in each portfolio													
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	5	3	4	4	3	3	6	7	5	1	1
3	2	2	2	4	2	2	3	2	4	5	6	5	1	1
4	2	3	2	5	3	5	6	3	5	8	8	6	1	1
5	3	4	3	5	5	7	8	4	7	10	12	6	3	1
6	3	6	7	12	13	14	12	9	11	11	11	7	2	1
7	1	1	2	5	4	4	6	5	4	5	4	1	1	1
8	2	4	5	10	10	9	13	13	12	16	17	5	14	1
9	1	6	7	9	7	9	8	12	7	10	8	2	22	1
10	1	2	4	6	6	6	9	10	12	11	10	14	2	1
11	2	3	4	6	10	6	12	10	11	15	16	13	4	1
12	1	1	2	4	4	5	6	8	7	10	11	8	4	1
13	2	2	3	5	7	9	8	9	10	13	18	7	14	1
14	5	6	9	10	15	17	15	12	15	20	20	22	18	1
15	4	5	5	9	9	12	8	12	12	17	14	24	10	1
16	6	4	5	6	7	9	6	5	7	11	10	14	3	1
17	6	6	8	6	10	12	8	9	9	11	13	22	24	1
18	6	6	7	7	9	12	10	10	14	17	18	35	31	3
19	1	4	2	11	10	9	9	13	15	20	15	19	3	1
20	1	2	2	5	6	7	8	11	8	11	12	5	2	1

Average number of securities

-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
3	4	4	7	7	8	8	8	9	11	12	11	8	1

Maximal number of securities

6	6	9	12	15	17	15	13	15	20	20	35	31	3

Minimal number of securities

1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 5. *Computational time range and the solution results for* δ

r^{VaR}	δ	MIP simplex iterations	branch-and-bound nodes	CPU	GAP [%]
−10.00	0.067711	15,104	805	43.1	–
−7.50	0.071381	3,182,319	441,196	10,800.0	1.85
−5.00	0.077699	3,169,107	195,210	10,800.1	8.72
−2.50	0.089562	1,668,609	116,278	10,800.1	29.31
−2.00	0.098133	1,554,694	76,373	10,800.2	32.57
−1.75	0.097387	1,177,463	69,993	10,800.3	36.15
−1.50	0.100295	1,162,975	65,026	10,800.3	38.61
−1.25	0.101089	1,713,553	71,822	10,800.4	39.98
−1.00	0.099639	2,744,317	95,456	10,800.1	42.44
−0.75	0.124678	2,730,531	101,598	10,800.1	47.37
−0.50	0.127738	2,824,989	84,196	10,800.4	51.10
−0.25	0.307067	3,022,921	76,426	10,800.1	65.44
−0.05	0.344089	3,027,166	71,625	10,800.1	67.75
0.00	0.369005	3,561,349	84,385	10,800.0	71.55

Table 6 presents optimization problem size after presolving.

Table 6. *Problem size after presolving*

r^{VaR}	Size of adjusted problem				Eliminated by presolving	
	constraints	variables	binary variables	linear variables	constraints	variables
−10.00	15,509	12,305	7,865	4,440	19,344	87,396
−7.50	15,722	12,416	7,976	4,440	19,131	87,285
−5.00	15,927	12,526	8,086	4,440	18,926	87,175
−2.50	16,180	12,655	8,215	4,440	18,673	87,046
−2.00	16,203	12,667	8,227	4,440	18,650	87,034
−1.75	16,216	12,674	8,234	4,440	18,637	87,027
−1.50	16,238	12,685	8,245	4,440	18,615	87,016
−1.25	16,261	12,698	8,258	4,440	18,592	87,003
−1.00	16,280	12,708	8,268	4,440	18,573	86,993
−0.75	16,308	12,722	8,282	4,440	18,545	86,979
−0.50	16,333	12,735	8,295	4,440	18,520	86,966
−0.25	16,353	12,745	8,305	4,440	18,500	86,956
−0.05	16,353	12,745	8,305	4,440	18,500	86,956
0.00	16,360	12,866	8,426	4,440	18,493	86,835

In the computational experiments the dataset of daily quotations from the Warsaw Stock Exchange were used. The computational time for the optimization model depends strongly on return Value-at-Risk parameter. The bigger value of r^{VaR} is set, the more CPU seconds is required for proving optimality. The computed values of expected portfolio return and the probability that return of investment is not less than r^{VaR} also depend on r^{VaR}. Increasing the value of r^{VaR} results in decreasing of expected portfolio return and increasing of probability that the return of investment is not less than r^{VaR}. Number of securities in obtained portfolios varies between one and thirty five.

6. CONCLUSIONS

In this paper a bi-objective portfolio selection by mixed integer programming has been proposed. The considered problem is based on a dynamic model of investment, in which the investor buys and sells securities in successive investment periods. The problem objective is to dynamically allocate the wealth on different securities to optimize by reference point method the portfolio expected return and the probability that the return is not less than a required level. The model incorporates dynamic balance constraints that allow the short-selling variables to take on non-negative values.

The computational experiments modeled on a real data from the Warsaw Stock Exchange have indicated that the approach is capable of finding optimal solutions for medium size problems in a reasonable computation time using commercially available software for mixed integer programming. The total computation time ranges from minutes to hours depending on the number of historical quotations in the optimization problem.

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