A Bi-Objective Portfolio Optimization
with Conditional Value-at-Risk

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Abstract. This paper presents a bi-objective portfolio model with the expected return as a performance measure and the expected worst-case return as a risk measure. The problems are formulated as a bi-objective linear program. Numerical examples based on 1000, 3500 and 4020 historical daily input data from the Warsaw Stock Exchange are presented and selected computational results are provided. The computational experiments prove that the proposed linear programming approach provides the decision maker with a simple tool for evaluating the relationship between the expected and the worst-case portfolio return.

Keywords: multi-criteria decision making, portfolio optimization, conditional value-at-risk, weighting approach, linear programming.

Mathematics Subject Classification: 91G10 – portfolio theory, 90C29 – multi-objective and goal programming, 90C90 – applications of mathematical programming, 90C05 – linear programming.

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1. INTRODUCTION

The portfolio problem, which involves computing the proportion of the initial budget that should be allocated in the available securities, is at the core of the field of financial management. A fundamental answer to this problem was given by Markowitz (1952, 1959, 1991, 1997) who proposed the mean-variance model which laid the basis of modern portfolio theory. In Markowitz's approach the problem is formulated as an optimization problem involving two criteria: the reward of portfolio, which is measured by the mean or expected value of return that should be maximized, and the risk of the portfolio, which is measured by the variance of return that should be minimized. In the presence of two criteria there is not a single optimal solution (portfolio structure), but a set of optimal portfolios, the so-called efficient portfolios, which trade-off between risk and return. Since the mean-variance theory of Markowitz, an enormous amount of papers have been published extending or modifying the basic model in three directions.

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The first path goes to simplification of the type and the amount of input data. The second direction concentrates on the introduction of the alternative measure of risk. Finally, the third involves the incorporation of additional criteria and/or constraints (e.g. Anagnostopoulos et al. (2010)).

The overall process of selecting a portfolio is divided into two stages (Markowitz, 1952). The first stage starts with observation, experience and ends with beliefs about the future performances of available securities. The second stage starts with relevant beliefs about future performances and ends with the choice of portfolio. One type of rule concerning choice of portfolio is that the investor should maximize the capitalized value of future returns. The decision maker places all his funds in the security with the greatest discounted value. Investor diversify his founds among all those securities which give maximum expected return. If two or more securities have the same value, then any of these or any combination of these is as good as any other. However, the portfolio with maximum expected return is not necessarily the one with minimum risk. The law of large numbers will insure that the actual yield of the portfolio will be almost the same as the expected yield. There is a rate at which the investor can gain expected return by taking on risk measure, or reducing risk by giving up expected return (Ogryczak, 2000). In the classical Markowitz model future returns are random variables that can be controlled by the two parameters: a portfolio's efficiency calculated by the expectation, and a risk, which is measured with variance. The classical problem is formulated as a quadratic program with continuous variables and some side constraints. Bai et al. (2009a, 2009b) have developed a new bootstrap-corrected estimator of the optimal return for the Markowitz mean-variance optimization. Markowitz and van Dijk (2003) find that under certain conditions, the single-period mean-variance model provides a good approximation to multi-period expected utility maximization.

Although the original Markowitz model forms a quadratic programming problem, following Sharpe (1971), many attempts have been made to linearize the portfolio optimization procedure (for instance Speranza (1993)). The linear program solvability is very important for applications to real-life financial and other decisions where the constructed portfolios have to meet numerous side constraints. The examples of them are minimum transaction lots, transaction costs or mutual funds characteristics etc. The introduction of these features leads to mixed integer programming problems.

For some basic investment decision-making approaches, the decision maker may be restricted to choosing only one of a discrete number of alternatives. For other scenarios, a diversified portfolio comprised of a convex combination of two or more alternatives may be feasible and will often better balance risk and return. Sharpe (1971) stated that "*if the essence of the portfolio analysis problem could be adequately captured in a form suitable for linear programming methods, the prospect for application would be greatly enhanced*". Linear programming efforts the decision maker the opportunity to determine an optimal balance between risk and return for modeling portfolio optimization problems with diversification among alternatives.

There is a vast literature on portfolio selection devoted to the balancing of risk and return in financial markets. The most celebrated of these (as it was previously written) is the approach of Markowitz (1952, 1959, 1991, 1997) where a quadratic

mean-variance model with risk measured by the covariance matrix of returns was developed. Konno and Yamazaki (1991) noted that the derivation of the covariance matrix can be cumbersome, attempting to solve a quadratic model has computational limitations in practice, and the optimal solution may consist of purchasing a large number of securities. They suggest employing linear objectives to alleviate these computational limitations. In spite of the fact that Sharpe (1963) developed a methodology for practical solution of the quadratic objective, many approaches have been taken to linearize the model. Sharpe (1967, 1971) and Stone (1973) both showed how the quadratic model could be transformed to an equivalent model with a separable quadratic function making it much easier to solve with linear approximation approaches. Leung and Wong (2008) have developed a multivariate Sharpe ratio statistic to test the hypothesis of the equality of multiple Sharpe ratios.

Biglova at al. (2004) identified several other criteria for estimating portfolio theory risk that can be employed in LP models instead of the covariance risk measure of Markowitz (1952, 1959, 1991, 1997). Among these include Gini's mean absolute difference as incorporated by Yitzhaki (1982) resulting in a LP for constructing efficient portfolios. In their linear optimization model, Konno (1990) and Konno and Yamazaki (1991) employed absolute deviation rather than covariance to measure the risk. They solved a problem with 224 stocks over 60 months on a real-time basis and found results similar to that of the mean-variance model but requiring much less computational effort. Speranza (1993) generalized this approach using a risk function that is a linear combination of two semi-absolute deviations of return from the mean.

Ogryczak (2000) formulated and solved a multi-objective LP consisting of one objective for each time period and showed the mean-variance approach of Markowitz (1952, 1959, 1991, 1997), the absolute deviation approach of Konno and Yamazaki (1991), and the mini-max approach of Young (1998) to be special cases.

Young (1998) formulated an LP portfolio model for maximizing the minimum return to select a diversified portfolio based on historical returns data. He referred to the LP as a mini-max model because of its greater familiarity and this convention will be followed. The performance of the model was compared to other similar linear and nonlinear models and statistical analysis and simulation were employed to find that the mini-max approach outperformed the mean-variance approach with respect to mean square estimation error under the widely used log-normal distribution. He showed the mini-max modeling approach to be compatible with expected utility maximization and explored the incorporation of fixed transaction charges.

Cai at al. (2000) considered an objective of minimizing the expected absolute deviation of the future returns from their mean for several stocks and found that the problem could be solved analytically rather than solving a LP model. Similarly, Teo and Yang (2001) minimized the average of maximum individual risk over a number of time periods and the resulting optimization model was found to be solvable as a bi-objective piecewise LP problem. Benati (2003) replaced the covariance objective of Markowitz (1952, 1959, 1991, 1997) with the worst conditional expectation resulting in a LP and developed an efficient algorithm for practical solutions to real-world sized problems. Ding (2006) considered LP models for maximizing the minimum returns but without the constraint for a minimum required average return for the portfolio as

in Young (1998). For these simpler LP models he was able to develop optimal control policies for four cases of assumptions regarding evaluations (forecasts) for the potential returns. Gulpinar and Rustem (2007) proposed multiple alternative return and risk scenarios and developed a min-max algorithm to determine an optimal worst-case investment strategy.

Rockafellar and Uryasev (2000), Krokhmal et al. (2002), and Mansini et al. (2007) all focused upon minimizing conditional value-at-risk ($CVaR$) and developed LP models, properties, and solution approaches for this setting. Schrage (2001) devoted a chapter to portfolio optimization featuring a LP model to maximize the minimum return and another to minimize expected downside risk. His wide-ranging treatment of this topic also included approximations for the covariance matrix, inclusion of transaction costs, and inclusion of taxes for the Markowitz (1952, 1959, 1991, 1997) model as well as the value-at-risk (VaR) model and several deterministic equivalents of other stochastic optimization models. Alexander and Baptista (2002, 2004) incorporated VaR and $CVaR$ as constraints in the Markowitz (1952, 1959, 1991, 1997) model and found the conditional value-at-risk approach dominant for managing risk. Benati and Rizzi (2007) formulated an integer linear programming model with value-at-risk replacing the covariance for the objective and developed properties for which polynomial time algorithms exist. Mansini et al. (2003a, b) provided a systematic overview, discussion of properties, and a computational comparison for LP solvable models for portfolio selection.

Kahneman and Tversky (1979) who thoroughly examine under-weighting and over-weighting of probabilities as key issues that may make insurance against losses attractive. Approaches for instance modeling began with Leland (1980) and Brennan and Solanki (1981) who examine maximizing expected utility subject to a budget constraint. But investor's preferences and probability belief may be difficult to ascertain and analyze so Aliprantis et al. (2000) introduced a LP approach to minimize the cost of a portfolio subject to a minimal payoff. Katsikis (2007) further refined computational approaches for this model. Aliprantis et al. (2002) extended the LP model by taking advantage of the situation where portfolio dominance information is also available.

Gilboa and Schmeidler (1989) employed a set of multiple prior probability distribution to model situation where the decision maker has too little information to discern a single prior distribution and expressed investor preferences as a utility function over this set. Chateauneuf et al. (2005) developed theoretical underpinnings for a number of important applications of the multiple priors. Gajdos et al. (2004) introduced a partial order on the set of multiple priors based on a reference prior distribution within the set termed an anchor. They proceeded to show that a decision maker who is averse to information imprecisions tends to maximize the minimum expected utility with respect to a subset of the multiple priors. Garlappi et al. (2007) employed confidence intervals around estimated expected returns to reflect decision making under multiple prior and modeled ambiguity aversion in terms of minimization of a function over these priors.

Sawik (2008) constructed the three stage lexicographic approach and the corresponding mixed integer programming formulations for the multi-criteria portfolio

optimization problem. The primary objective is to maximize expected portfolio return, then the minimization of risk probability of portfolio loss versus the maximization of amount of capital to be invested in portfolio is considered, and finally the minimization of number of stocks in optimal portfolio is achieved. Some additional examples of the portfolio multi-criteria mixed integer programming formulations with use of the VaR can be found in Sawik (2009a, 2009b, 2010).

Polak et al. (2010) constructed mini-max portfolio model with linear programming approach. Employed risk measured as the worst-case return and a portfolio from maximizing returns subject to a risk threshold. They proceeded to show parametric analysis of the risk threshold connected their model to expected value along a continuum, revealing an efficient frontier segmenting investors by risk preferences.

Chen and Kwon (2010) develop a robust portfolio selection model for tracking a market index using a subset of its assets. The model is a binary program that seeks to maximize similarity between selected assets and the assets of the target index. Presented optimization model allows uncertainty in the objective function by using a computationally tractable robust framework that can control the conservativeness of the solution. This protects against worst-case realizations of potential estimation errors and other deviations.

2. PERCENTILE MEASURES OF RISK IN THE LITERATURE

In the mean-variance context of Markowitz (1952, 1959, 1991, 1997), the variance or standard deviation is adopted to measure the risk exposure of financial portfolios (see e.g. Bai et al., 2009a, 2009b; Leung and Wong, 2008). However, these measures fail to capture the downside risk. To circumvent this problem, many academics have proposed value-at-risk (VaR) (Jourion, 2006) and the conditional value-at-risk ($CVaR$) (Alexander and Baptista, 2002, 2004; Sarykalin et al., 2008).

Generally risk management has received much attention from practitioners and regulators as well as academics in the last few years, with value-at-risk emerging as one of the most popular tools. Jourion (2006), Linsmeier and Pearson (2000), Alexander and Baptista (2002, 2004), Hull (2003), and Chance (2004) note that value-at-risk is widely used as a risk management tool by corporate treasurers, dealers, fund managers, financial institutions, and regulators (see Basel Committee on Banking Supervision (1996, 2003)).

In contrast, some researchers have extensively criticized the use of value-at-risk as a measure of risk. For instance Artzner et al. (1999) pointed out that value-at-risk is not a coherent measure of risk since it fails to hold the sub-additivity property. Moreover, value-at-risk does not explain the magnitude of the loss when the value-at-risk limit is exceeded. Furthermore, it is difficult to optimize when using calculated scenarios, and this leads to the use of an alternative measure, which is conditional value-at-risk.

Basak and Shapiro (2001) show that when an agent faces a value-at-risk constraint at the initial date in a continuous-time model, the agent may select a larger exposure to risky securities than he or she would have chosen in its absence. Yiu (2004) shows that imposing a dynamic constraint in a continuous-time model leads

an agent to select a smaller exposure to risky stocks than it would have been chosen in case of its absence. This reasons incline mentioned previously researchers to proposed use of conditional value-at-risk rather than value-at-risk. Pflug (2000) proved that conditional value-at-risk is a consistent measure of risk for its sub-additivity and convexity properties. Uryasev (2000) presented a description of both: (a) an approach for minimizing conditional value-at-risk and (b) optimization problems with conditional value-at-risk constraints. Alexander and Baptista (2004) noticed that the presence of a value-at-risk constraint will cause a slightly risk-averse agent to select a portfolio that has a smaller standard deviation than the one that would have been selected in its absence. However, there are also conditions under which the constraint causes a highly risk-averse agent to select a portfolio that has a larger standard deviation. Conditional value-at-risk constraint is tighter than a value-at-risk constraint when the conditional value-at-risk and value-at-risk bounds coincide, these portfolio choice results are also true and to a greater extent if a conditional value-at-risk constraint is imposed. Therefore, a conditional value-at-risk constraint is more effective than a value-at-risk constraint as a tool to control slightly risk-averse agents, but has a more perverse effect on highly risk-averse agents.

However, this perverse result weakens or even disappears when a risk-free security is available, or the conditional value-at-risk bound is larger than the value-at-risk bound. Moreover, if the conditional value-at-risk bound is set at a level so that conditional value-at-risk constraint has the same perverse effect on highly risk-averse agents as the value-at-risk constraints, then the conditional value-at-risk constraint will result in slightly risk-averse agents selecting portfolios with small standard deviations than those when a value-at-risk constraint is imposed. If the conditional value-at-risk bound is set at an even larger level so that the conditional value-at-risk constraint decreases the standard deviations of the optimal portfolios of slightly risk-averse agents to select portfolios with smaller standard deviations than those when a value-at-risk constraint is imposed. Hence, when the conditional value-at-risk bound is set between these two levels, a conditional value-at-risk constraint dominates a value-at-risk constraints as a risk management tool.

The proposed multi-criteria portfolio approach allows aforementioned two percentile measures of risk in financial engineering: value-at-risk and conditional value-at-risk to be applied for managing the risk of portfolio loss. The proposed mixed integer and linear programming models provide the decision maker with a simple tool for evaluating the relationship between expected and worst-case loss of portfolio return. A risk measure can be linear program computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios. VaR and $CVaR$ have been widely used in financial engineering in the field of portfolio management (e.g. Sarykalin et al. (2008)). $CVaR$ is used in conjunction with VaR and is applied for estimating the risk with non-symmetric cost distributions. Uryasev (2000) and Rockafellar and Uryasev (2000, 2002) introduced a new approach to select a portfolio with the reduced risk of high losses. The portfolio is optimized by calculating VaR and minimizing $CVaR$ simultaneously. Polak et al. (2010) noticed that objectives such as minimizing variation or the popular value-at-risk objective may be quite effective especially during periods of slow or moderate economic changes.

3. DEFINITIONS OF PERCENTILE MEASURES OF RISK

Let $\alpha \in (0,1)$ be the confidence level.

The percentile measures of risk, VaR and $CVaR$ can be defined as below:

– Value-at-Risk (VaR) at a $100\alpha\%$ confidence level is the targeted return of the portfolio such that for $100\alpha\%$ of outcomess the return will not exceed VaR. In other words, VaR is a decision variable based on the α-percentile of return, i.e., in $100(1-\alpha)\%$ of outcomes the return may not attain VaR.

– Conditional Value-at-Risk ($CVaR$) at a $100\alpha\%$ confidence level is the expected return of the portfolio in the worst $100(1-\alpha)\%$ of the cases. Allowing $100(1-\alpha)\%$ of the outcomes not exceed VaR, and the mean value of these outcomes is represented by $CVaR$.

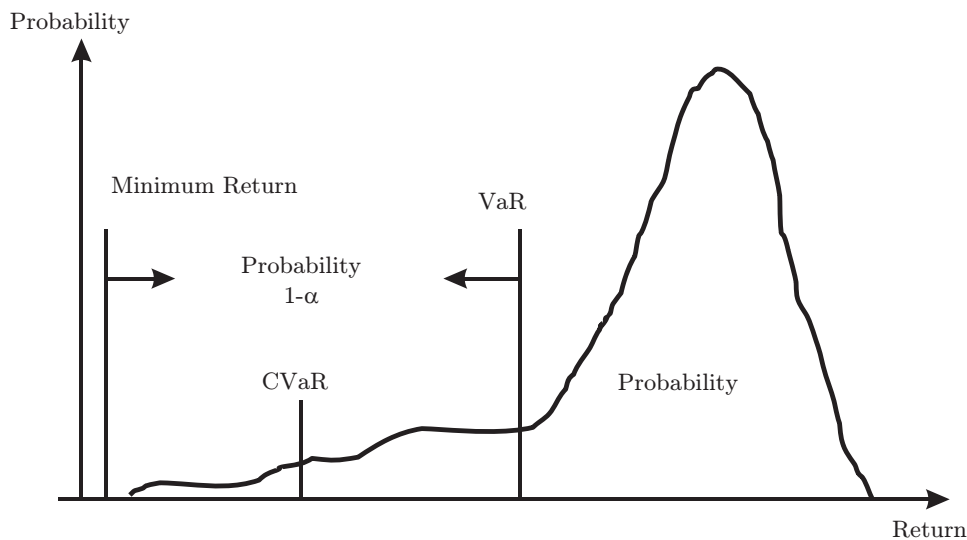Figure 1 illustrates value-at-risk and conditional value-at-risk for a given portfolio and the confidence level α.

Fig. 1. *Value-at-Risk and Conditional Value-at-Risk*

Value-at-risk represents the minimum return associated with a specified confidence level of outcomes (i.e., the likelihood that a given portfolio's return will not be less than the amount defined as VaR). However, VaR does not account for properties of the cost distribution beyond the confidence level and hence does not explain the magnitude of the return when the VaR limit is not exceeded. On the other hand, $CVaR$ (conditional value-at-risk) focuses on the tail of the return distribution, that is, on outcomes with the lowest return.

Since VaR and $CVaR$ measure different parts of the cost distribution, VaR may be better for optimizing portfolios when good models for tails are not available, otherwise $CVaR$ may be preferred, e.g. (Rockafellar, 2000, 2002; Sarykalin, 2008; Uryasev, 2000). When using $CVaR$ to minimize worst-case costs, $CVaR$ is always less than VaR.

4. PROBLEM FORMULATION

This section presents mathematical programming formulations **CV** and **E_CV**, respectively for the single- and bi-objective selection of optimal portfolio. Table 1 shows the notation used in this section.

Table 1. *Notation*

<table>
<tr><td colspan="2" align="center">Indices</td></tr>
<tr><td>i</td><td>= historical time period i, $i \in M = \{1, \ldots, m\}$ (day)</td></tr>
<tr><td>j</td><td>= security j, $j \in N = \{1, \ldots, n\}$</td></tr>
<tr><td colspan="2" align="center">Input Parameters</td></tr>
<tr><td>r_{ij}</td><td>= observed return of security j in historical time period i</td></tr>
<tr><td>p_i</td><td>= probability of historical portfolio realization i</td></tr>
<tr><td>α</td><td>= confidence level</td></tr>
<tr><td>λ</td><td>= weight in the objective function of model E_CV</td></tr>
<tr><td colspan="2" align="center">Decision Variables</td></tr>
<tr><td>R_i</td><td>= tail return, i.e., the amount by which VaR exceeds returns in scenario i</td></tr>
<tr><td>VaR</td><td>= Value-at-risk of portfolio return based on the α-percentile of return, i.e., in $100\alpha\%$ of historical portfolio realization, the outcome must be greater than VaR</td></tr>
<tr><td>x_j</td><td>= amount of capital invested in security j</td></tr>
</table>

In the risk averse objective (1), VaR is a decision variable denoting the Value-at-risk. Let α be the confidence level for the total portfolio return distribution across all historical portfolios. It can also be said that VaR is a decision variable based on the α-percentile of portfolio return, for instance, in $100(1 - \alpha)\%$ of the historical portfolios, the outcome will not exceed VaR.

Finally, $CVaR$ is a weighted measure of VaR and the portfolio return below VaR (which, again, could be an extremely small return or actually losses). R_i is defined as the tail portfolio return for scenario (historic portfolio) i, i.e., the amount by which VaR exceeds the average return in portfolio i.

Model CV

Maximize

$$VaR - (1 - \alpha)^{-1} \sum_{i \in M} p_i R_i \tag{1}$$

subject to

$$\sum_{j \in N} x_j = 1 \tag{2}$$

$$R_i \geqslant VaR - \sum_{j \in N} r_{ij} x_j; i \in M \tag{3}$$

$$x_j = 0; j \in N : \sum_{i \in M} p_i r_{ij} \leqslant 0 \tag{4}$$

$$x_j \geqslant 0; j \in N \tag{5}$$

$$R_i \geqslant 0; i \in M \tag{6}$$

Constraint (2) ensures that all capital is invested in the portfolio (the selected securities). Risk constraint (3) defines the tail losses for scenario i. Constraint (4) eliminates from selection the assets with non-positive expected return over all scenarios. Eq. (5) and (6) are non-negativity conditions.

A risk-aversive decision maker wants to maximize the Conditional Value-at-Risk, as shown in the objective functions of models **CV** and **E_CV**. Note that as R_i is constrained of being positive, the models tries to increase VaR and hence positively impact the objective functions. However, large increases in VaR may result in more historic portfolios (scenarios) with tail returns, counterbalancing this effect.

The portfolio model **E_CV** presented below provides flexibility in how a decision maker wants to balance its risk tolerance with the expected portfolio returns.

Model E_CV

Maximize

$$\lambda(VaR - (1 - \alpha)^{-1} \sum_{i \in M} p_i R_i) + (1 - \lambda)(\sum_{i \in M} p_i (\sum_{j \in N} r_{ij} x_j) \tag{7}$$

subject to (2), (3), (4), (5), (6).

The nondominated solution set of the bi-criterion portfolio can be found by the parametrization on λ the weighted-sum program **E_CV**.

5. COMPUTATIONAL EXAMPLES

In this section the strength of $CVaR$ approach and LP models **CV** and **E_CV** is demonstrated on computational examples. The data sets for the example problems were based on historic daily portfolios of the Warsaw Stock Exchange ranging from 1000 days (from 29th of July 2005 to 30th of January 2009) and 135 securities, through 3500 days (from 30th April 1991 to 31st of January 2007) and 250 securities, to 4020 days (from 30th April 1991 to 30th of January 2009) and 240 securities.

In the computational experiments the five levels of the confidence level was applied $\alpha \in \{0.99, 0.95, 0.90, 0.75, 0.50\}$, and for the weighted sum program **E_CV** the subset of nondominated solutions were computed by parametrization on $\lambda \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

The results obtained for model **CV** are presented in Table 2 and in Figures 2 and 3. Table 2 shows the optimal values of $CVaR$, VaR, and the expected portfolio return for different confidence level α and the size of the input data set. In all cases the CPU time increases when the confidence level decreases. The number of securities in the optimal portfolios varies between 14 and 39 assets. The relation between the conditional value-at-risk $CVaR$ and the confidence level α is also shown in Figure 2. VaR and $CVaR$ increase as the confidence level decreases.

Figure 3 shows the number of securities in the computed portfolio and the computational time range for different size of input data set and model **CV**. The number of stocks selected for the optimal portfolio is independent on the confidence level and the size of input data set. The computational results for model **E_CV** are presented in Figures 4–9 and Table 3. The conditional value-at-risk is more negative than value-at-risk, which is clearly shown in Figures 4 and 5.
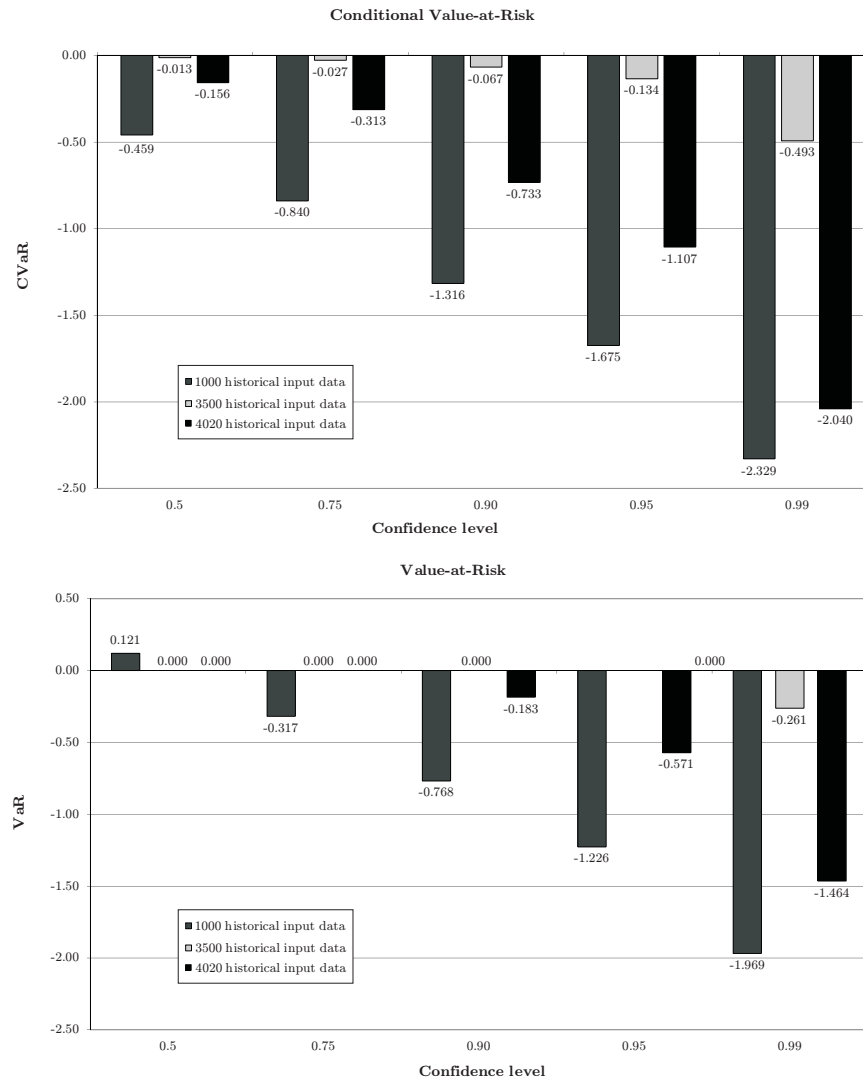

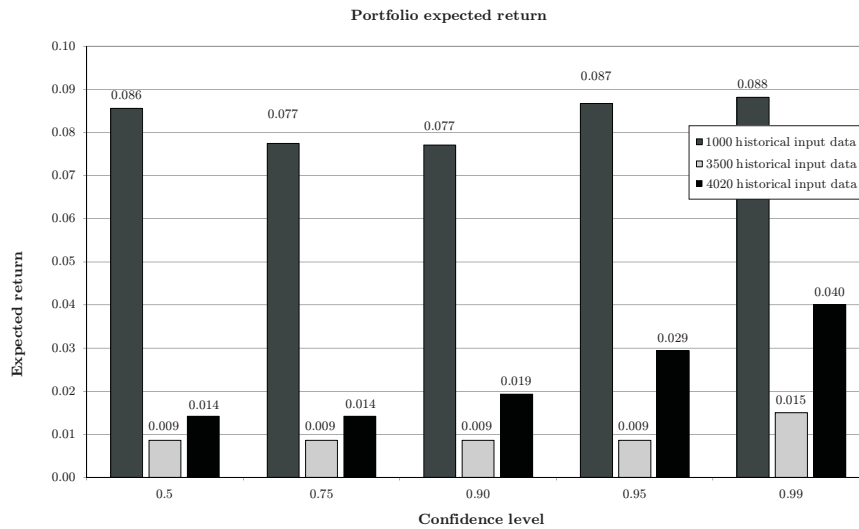**Fig. 2.** *Comparison of $CVaR$, VaR and the expected return for **CV** model with 1000, 3500 and 4020 historical input data*

Fig. 2. *(continued)*

Table 2. *Solutions results for* **CV** *model and 1000, 3500 and 4020 historical input data*

Confidence level α	0.99	0.95	0.90	0.75	0.50
1000 input data, Var.=1136, Cons.=1001, Nonz.=118100					
CVaR	-2.329320	-1.674800	-1.316210	-0.839905	-0.458828
VaR	-1.968690	-1.225590	-0.767942	-0.316816	0.120616
Expected portfolio return	0.088178	0.086781	0.077124	0.077459	0.085606
No. of securities in portfolio	16	25	30	30	39
Dual simplex iterations	226	506	948	1716	2202
CPU $^{(a)}$	0.124	0.202	0.358	0.686	0.951
$(1-\alpha)^{-1}\sum_{i=1}^{m}p_iR_i$	0.360628	0.449209	0.548270	0.523088	0.579444
No. of non-zero R_i	4	38	86	270	478
Confidence level α	0.99	0.95	0.90	0.75	0.50
3500 input data, Var.=3736, Cons.=3501, Nonz.=332550					
CVaR	-0.492633	-0.133975	-0.066987	-0.026795	-0.013398
VaR	-0.260505	0.000000	0.000000	0.000000	0.000000
Expected portfolio return	0.014985	0.008625	0.008625	0.008625	0.008625
No. of securities in portfolio	37	14	14	14	14
Dual simplex iterations	883	2065	2062	2800	4871
CPU $^{(a)}$	1.029	2.854	3.385	4.336	8.127
$(1-\alpha)^{-1}\sum_{i=1}^{m}p_iR_i$	0.232127	0.133975	0.066987	0.026795	0.013397
No. of non-zero R_i	20	56	56	56	56
Confidence level α	0.99	0.95	0.90	0.75	0.50
4020 input data, Var.=4189, Cons.=4021, Nonz.=349555					
CVaR	-2.039670	-1.106810	-0.733006	-0.312627	-0.156314
VaR	-1.464350	-0.570700	-0.183289	0.000000	0.000000
Expected portfolio return	0.040122	0.029419	0.019309	0.014214	0.014214
No. of securities in portfolio	23	34	39	27	27
Dual simplex iterations	582	2217	3414	6058	8019
CPU $^{(a)}$	0.702	2.730	3.993	9.048	12.854
$(1-\alpha)^{-1}\sum_{i=1}^{m}p_iR_i$	0.575319	0.536106	0.549717	0.312627	0.156314
No. of non-zero R_i	30	184	384	557	557

$^{(a)}$ CPU seconds for proving optimality on a laptop with Intel® Core 2 Duo T9300, 2.5GHz, RAM 4GB, CPLEX v.11.

Number of securities in portfolio
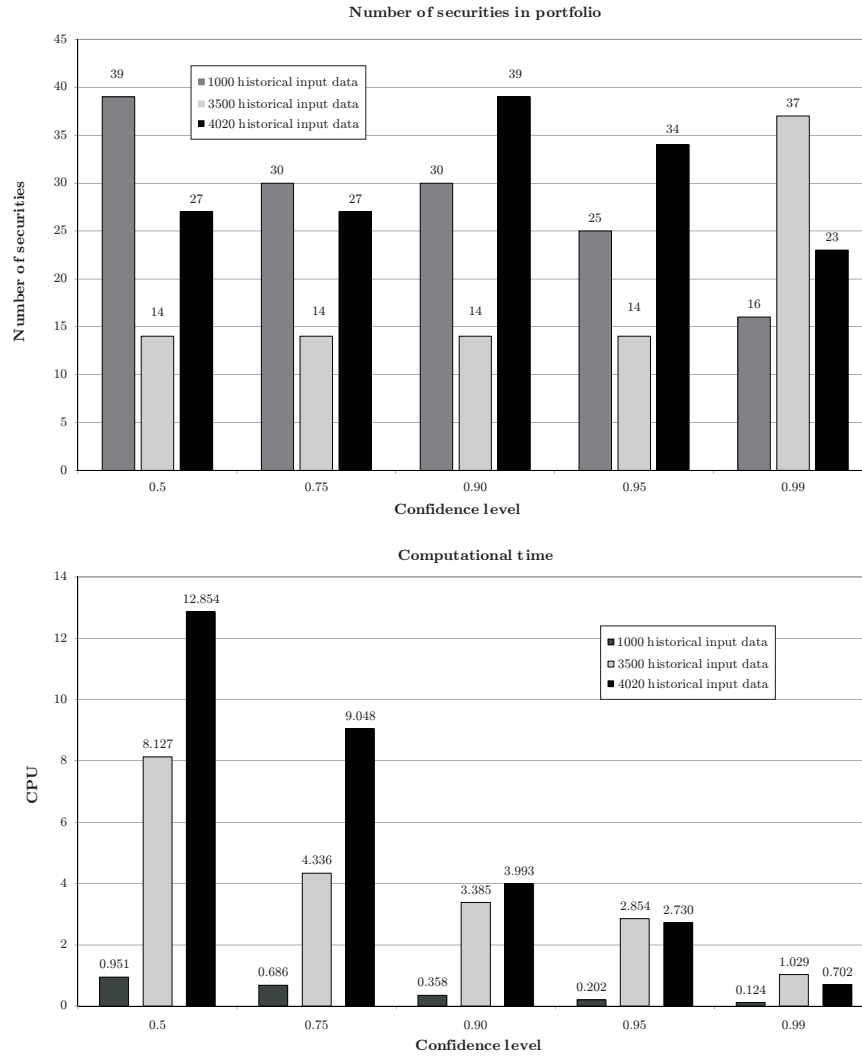

Computational time

Fig. 3. *Number of securities in an optimal portfolio and computational time range for* **CV** *model with 1000, 3500 and 4020 input data*

The confidence level α has a strong impact on obtained values $CVaR$ and VaR. The expected portfolio returns for different confidence level *alpha* and weight λ and for the three different historical portfolios are shown in Figure 6. Figure 7 shows that the number of securities selected for the optimal portfolio does not clearly depend either on the confidence level α, or on the size of the historical input data set. In weighted-sum program the weight λ only slightly influences the number of selected stock.

Figure 8 presents the efficient frontiers of the bi-objective model **E_CV** – conditional value-at-risk vs. expected return for the three different historical scenarios. The trade-off between the conditional value-at-risk and expected portfolio return is clearly shown as a concave efficient frontier. Figure 9 presents the CPU time required to obtain the optimal solution for **E_CV** model for different historical input data set. The computational time increases with the confidence level α and the size of input data set.

Conditional Value-at-Risk (1000 historical input data)

Conditional Value-at-Risk (3500 historical input data)

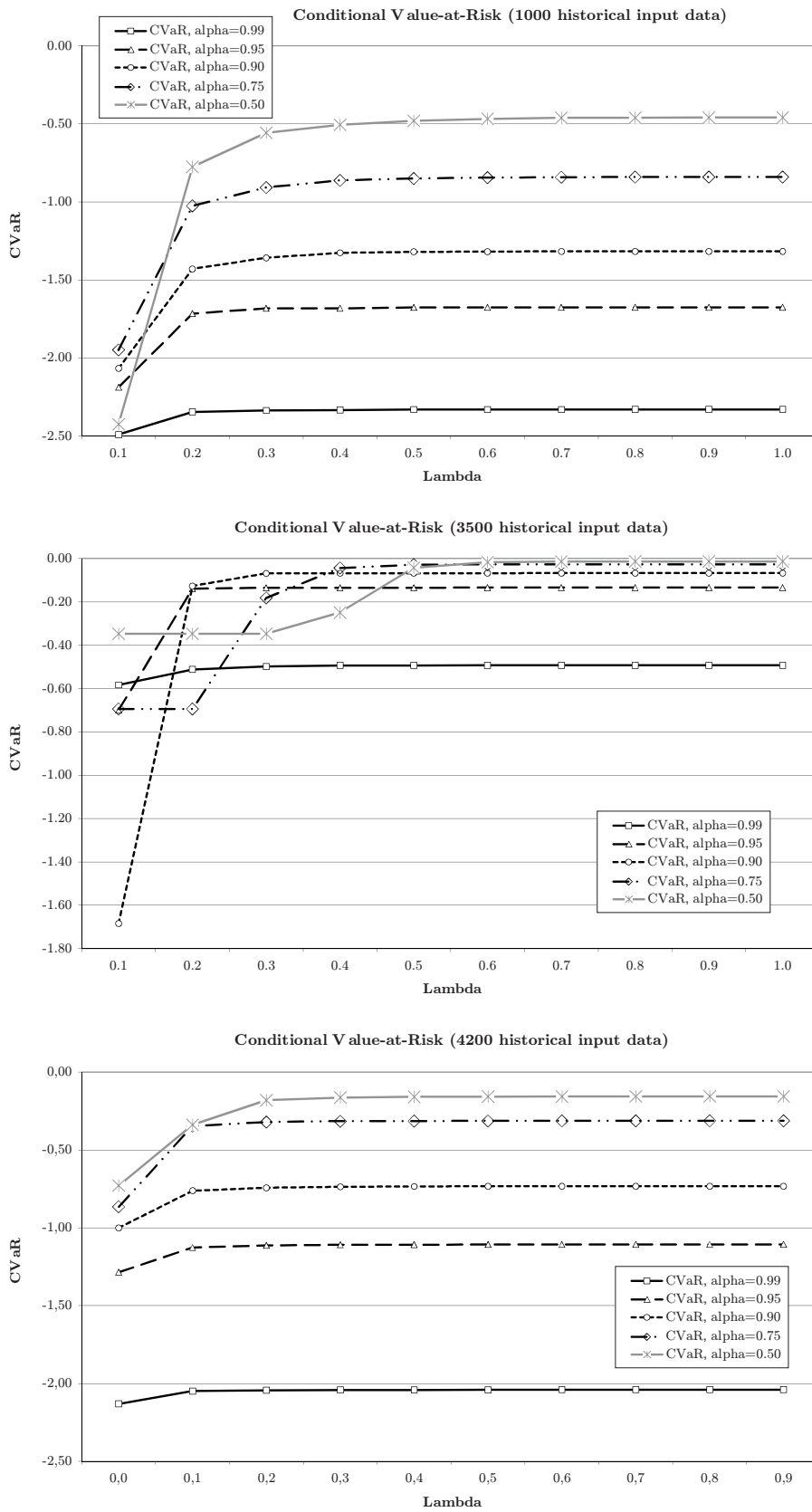**Conditional Value-at-Risk (4200 historical input data)**

Fig. 4. *Conditional Value-at-Risk for different confidence levels, and 1000, 3500 and 4020 historical input data – model* **E_CV**

Value-at-Risk (1000 historical input data)

Value-at-Risk (3500 historical input data)

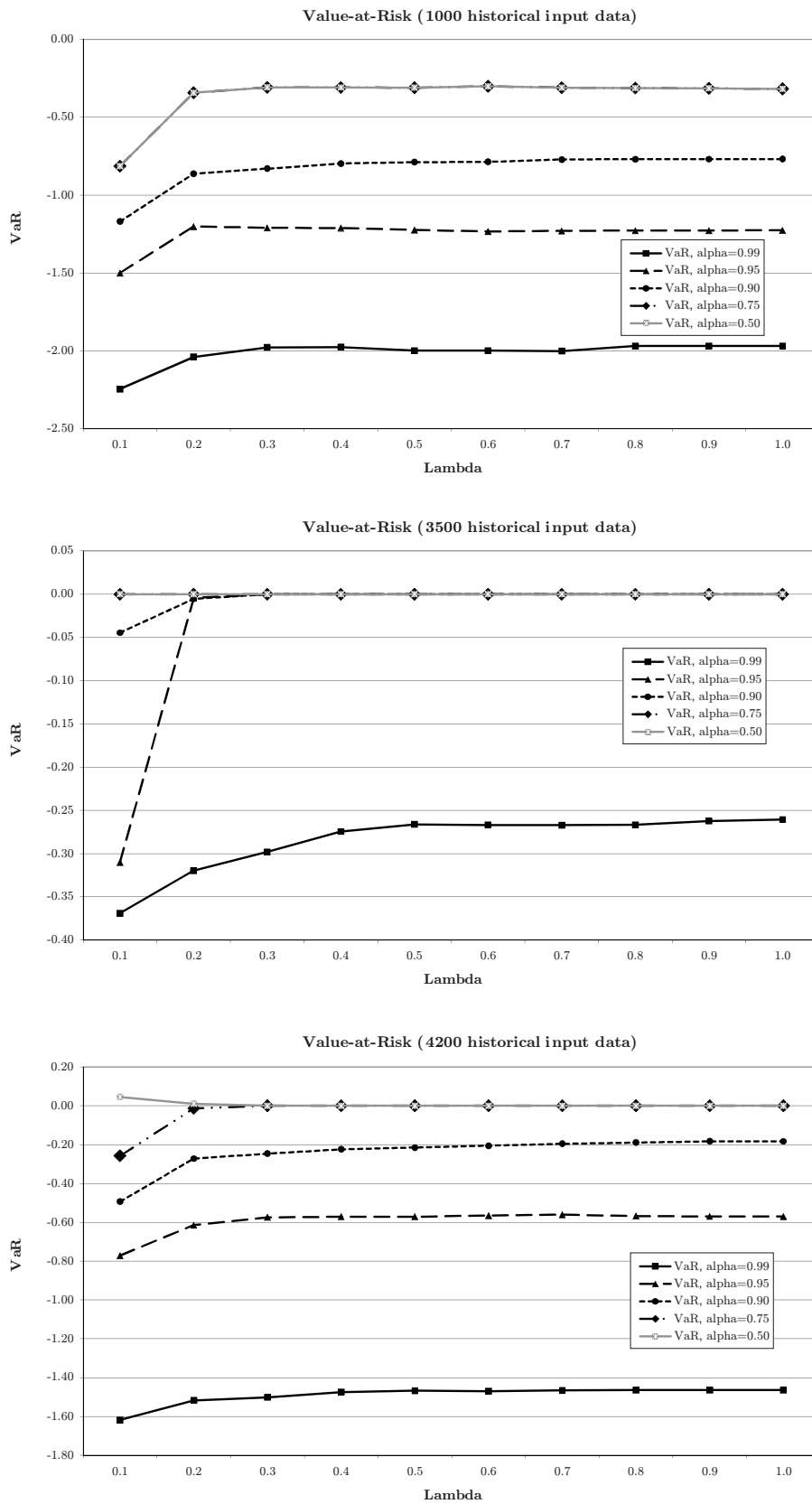**Value-at-Risk (4200 historical input data)**

Fig. 5. *Value-at-Risk for different confidence levels and 1000, 3500 and 4020 historical input data – model* **E_CV**

Expected portfolio return (1000 historical input data)

Expected portfolio return (3500 historical input data)

Expected portfolio return (4020 historical input data)

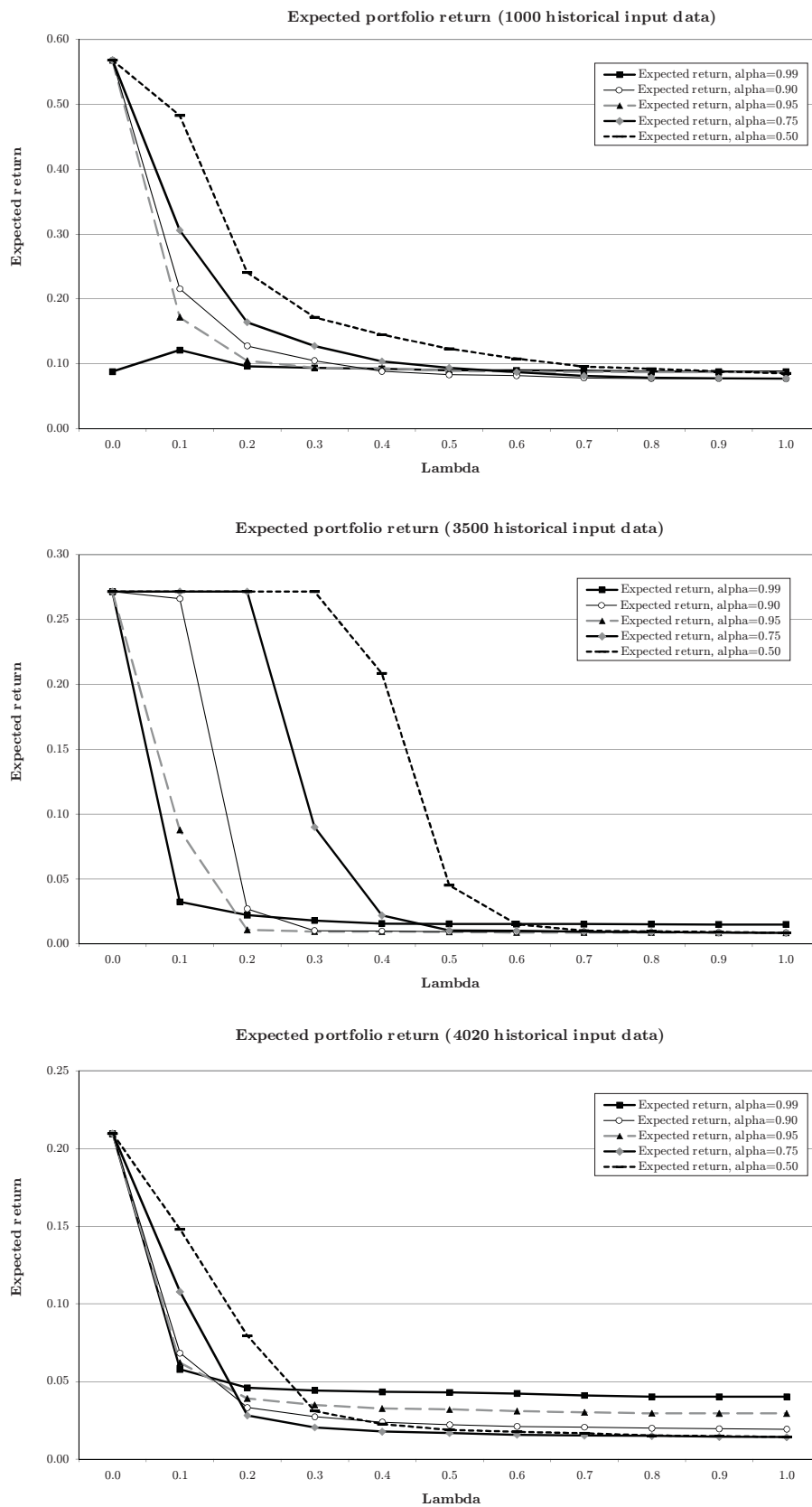**Fig. 6.** *Expected portfolio return for different confidence levels and 1000, 3500, 4020 historical input data – model* **E_CV**

Securities in optimal portfolio (1000 historical input data)

Securities in optimal portfolio (3500 historical input data)

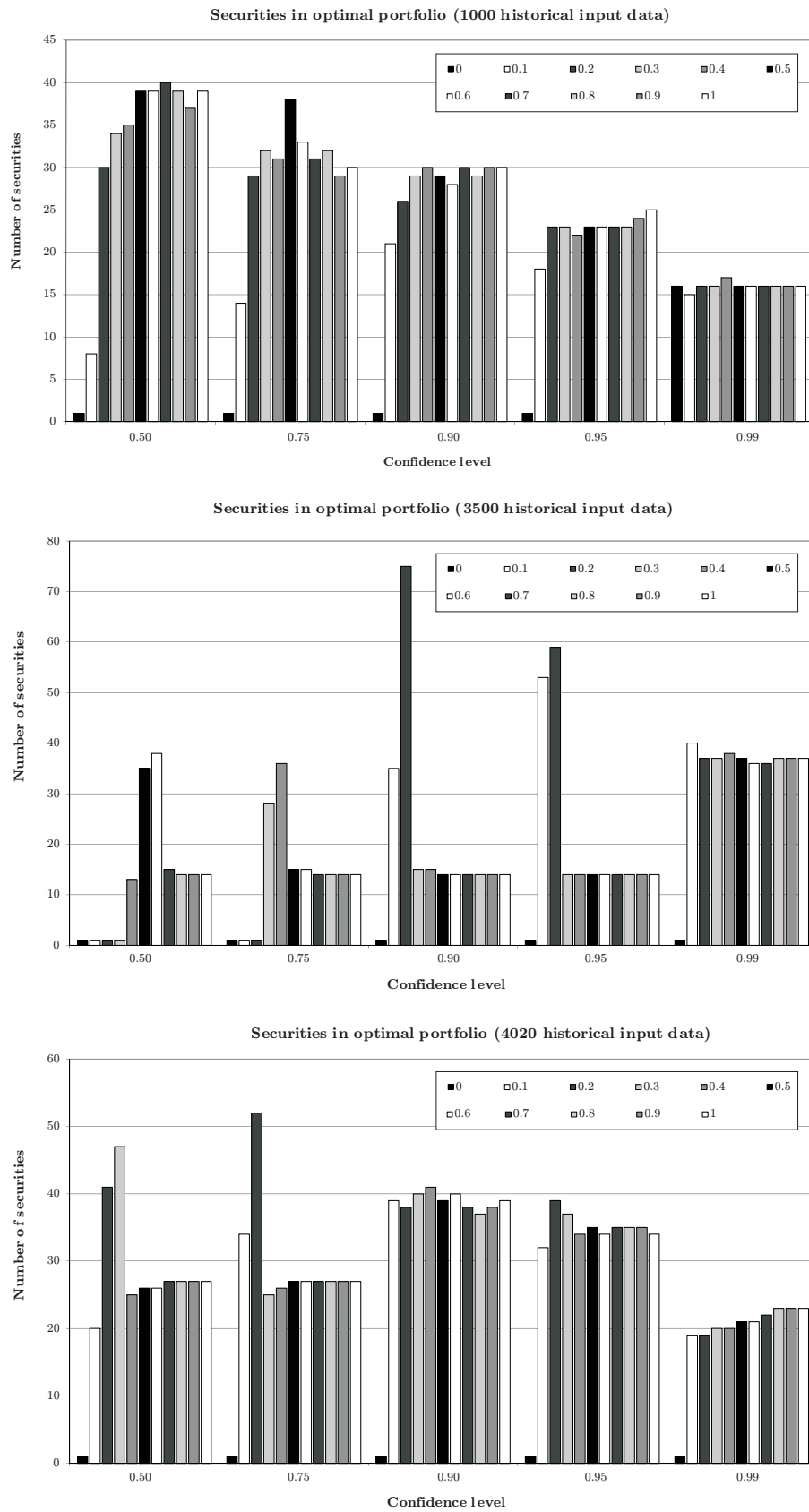**Securities in optimal portfolio (4020 historical input data)**

Fig. 7. *Number of assets in the optimal portfolio for 1000, 3500, 4020 historical input data – model* **E_CV**

Pareto frontier (1000 historical input data)

Pareto frontier (3500 historical input data)

Pareto frontier (4020 historical input data)
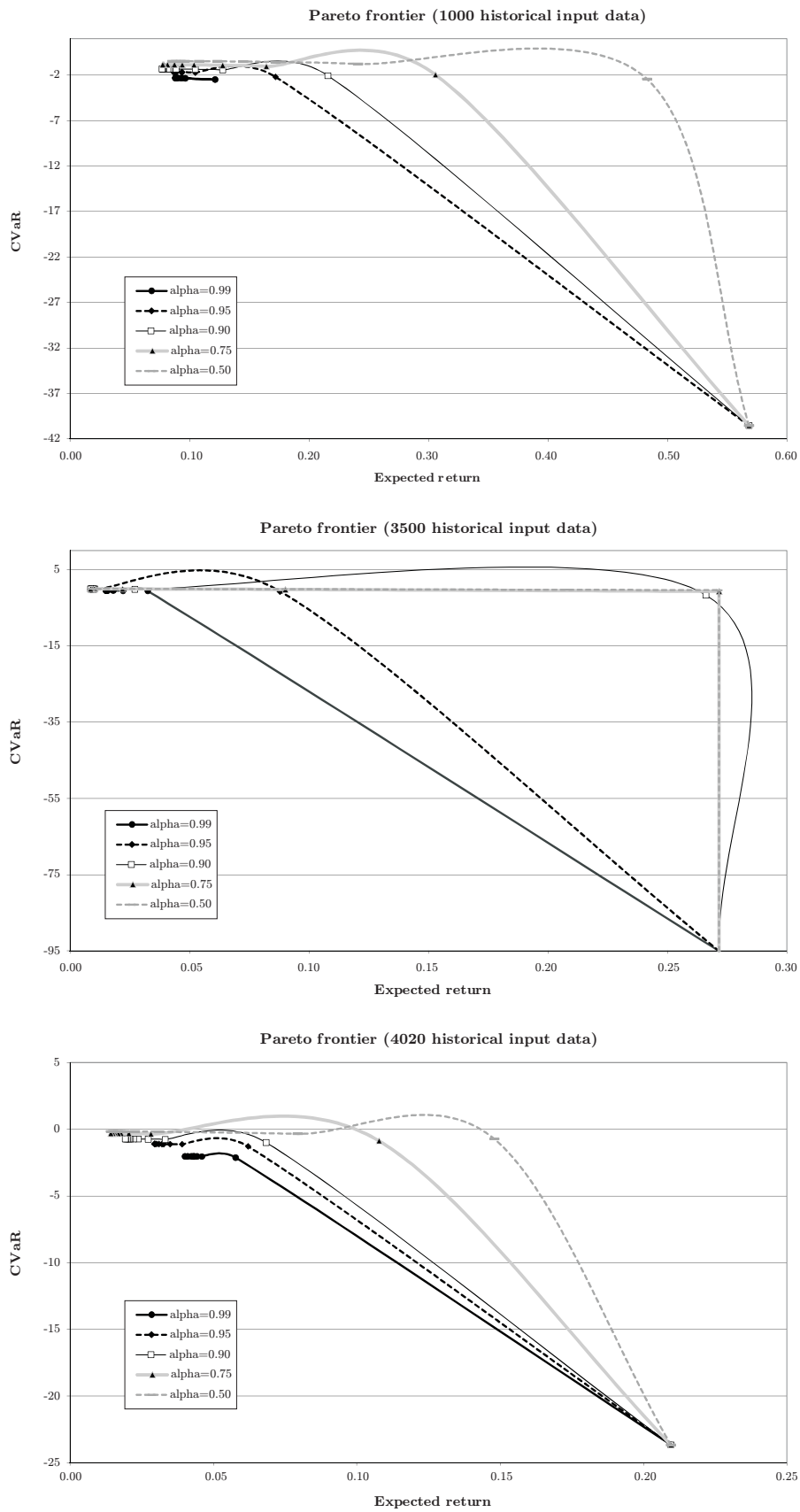
Fig. 8. *Pareto frontier for the bi-objective model* **E_CV** *for the three different scenario sizes*

Computational time (1000 historical input data)

Computational time (3500 historical input data)

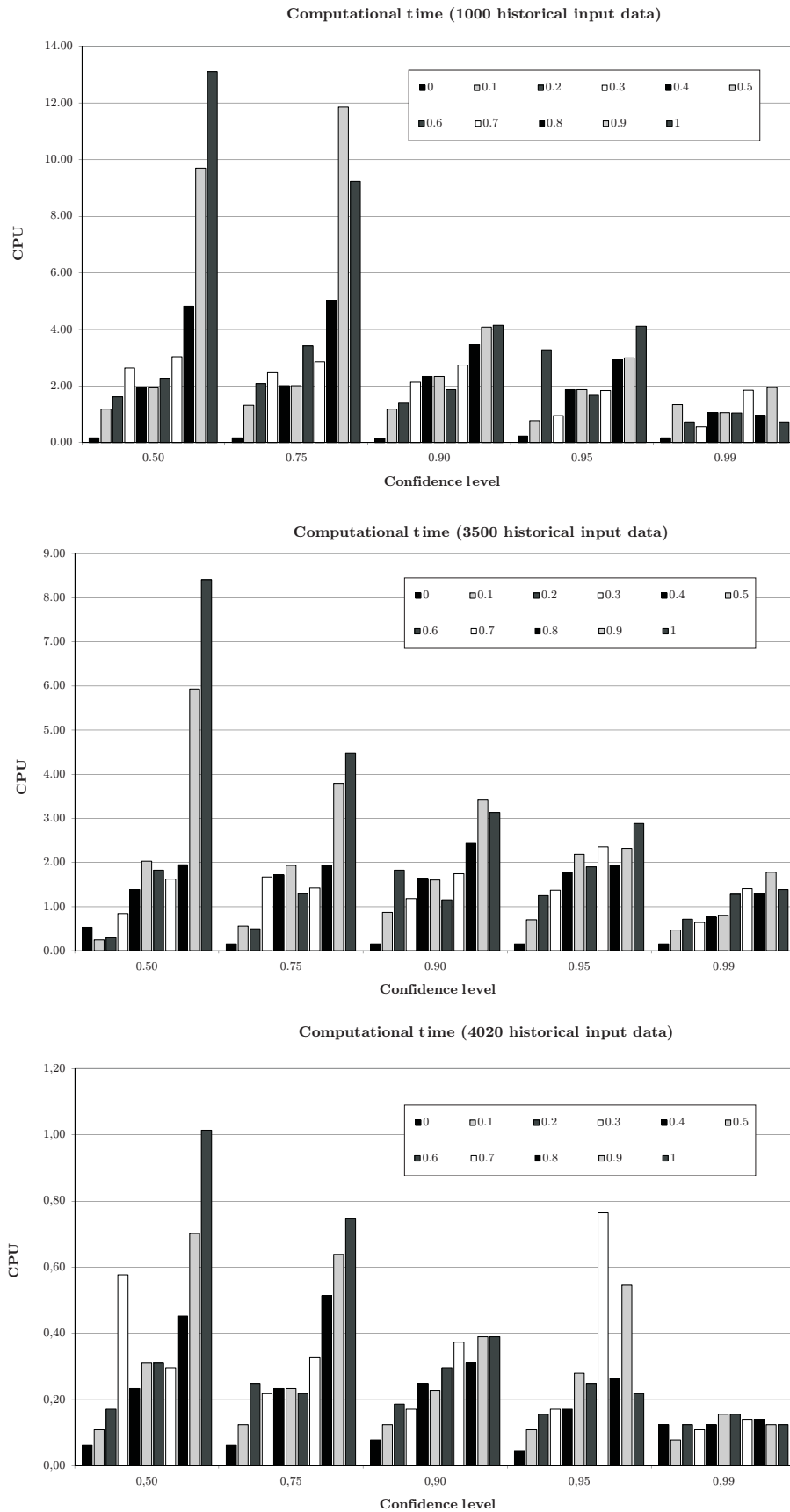**Computational time (4020 historical input data)**

Fig. 9. *Central processing unit – computational time for model* **E_CV** *and the three different historical scenarios*

The examples of nondominated solutions for model **E_CV** with different values of the confidence level *alpha* and for 4020 historic portfolios (scenarios) are presented in Table 3.

Table 3. *Nondominated solutions for the weighted-sum program* **E_CV** *for different confidence levels* α *and for 4020 historical input data*

λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α = 0.99											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-2.1309	-2.0491	-2.0437	-2.0420	-2.0414	-2.0408	-2.0401	-2.0397	-2.0397	-2.0397
VaR	-23.6607	-1.6186	-1.5172	-1.5021	-1.4748	-1.4680	-1.4699	-1.4664	-1.4644	-1.4644	-1.4644
Expected return	0.2096	0.0578	0.0459	0.0443	0.0434	0.0430	0.0423	0.0410	0.0401	0.0401	0.0401
No. of assets	1	19	19	20	20	21	21	22	23	23	23
Dual simplex iter.	0	412	609	518	640	730	636	668	693	739	582
CPU	0.173	1.341	0.733	0.561	0.748	1.060	1.045	1.856	0.967	1.950	0.733
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.5123	0.5320	0.5416	0.5672	0.5735	0.5709	0.5737	0.5753	0.5753	0.5753
No. of non-zero R_i	0	30	30	30	31	31	29	30	30	30	30
α = 0.95											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-1.2846	-1.1268	-1.1134	-1.1091	-1.1084	-1.1075	-1.1070	-1.1068	-1.1068	-1.1068
VaR	-23.6607	-0.7719	-0.6147	-0.5743	-0.5708	-0.5712	-0.5660	-0.5611	-0.5677	-0.5692	-0.5707
Expected return	0.2096	0.0621	0.0390	0.0349	0.0326	0.0320	0.0310	0.0301	0.0295	0.0295	0.0294
No. of assets	1	32	39	37	34	35	34	35	35	35	34
Dual simplex iter.	0	1232	1426	1388	1745	1797	1784	1780	2148	2013	2217
CPU	0.234	0.780	3.276	0.951	1.872	1.872	1.669	1.840	2.917	2.995	4.118
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.5127	0.5121	0.5391	0.5383	0.5372	0.5415	0.5459	0.5391	0.5376	0.5361
No. of non-zero R_i	0	183	182	184	185	185	186	182	185	186	180
α = 0.90											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-1.0003	-0.7620	-0.7430	-0.7365	-0.7344	-0.7336	-0.7333	-0.7331	-0.7330	-0.7330
VaR	-23.6607	-0.4936	-0.2721	-0.2461	-0.2238	-0.2152	-0.2057	-0.1954	-0.1890	-0.1828	-0.1833
Expected return	0.2096	0.0683	0.0332	0.0273	0.0238	0.0221	0.0211	0.0206	0.0200	0.0195	0.0193
No. of assets	1	39	38	40	41	39	40	38	37	38	39
Dual simplex iter.	0	1912	2204	2639	2378	2480	2526	2926	3278	3349	3414
CPU	0.156	1.185	1.404	2.137	1.669	2.340	1.874	2.745	3.463	4.087	4.149
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.5068	0.4899	0.4969	0.5127	0.5193	0.5279	0.5379	0.5441	0.5502	0.5497
No. of non-zero R_i	0	388	384	379	383	380	383	383	383	384	391
α = 0.75											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-0.8647	-0.3461	-0.3206	-0.3151	-0.3140	-0.3131	-0.3128	-0.3127	-0.3126	-0.3126
VaR	-23.6607	-0.2573	-0.0138	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Expected return	0.2096	0.1077	0.0282	0.0205	0.0178	0.0168	0.0157	0.0152	0.0150	0.0144	0.0142
No. of assets	1	34	52	25	26	27	27	27	27	27	27
Dual simplex iter.	0	3420	4191	4156	3904	3551	3961	4015	4885	6022	6058
CPU	0.171	1.326	2.090	2.496	2.683	2.012	3.416	2.854	5.023	11.856	9.235
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.6074	0.3323	0.3206	0.3151	0.3140	0.3131	0.3128	0.3127	0.3126	0.3126
No. of non-zero R_i	0	1089	723	554	596	549	453	489	553	478	557
α = 0.50											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-0.7286	-0.3386	-0.1802	-0.1631	-0.1586	-0.1574	-0.1569	-0.1564	-0.1564	-0.1563
VaR	-23.6607	0.0454	0.0107	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Expected return	0.2096	0.1480	0.0794	0.0310	0.0226	0.0189	0.0176	0.0166	0.0152	0.0149	0.0142
No. of assets	1	20	41	47	25	26	26	27	27	27	27
Dual simplex iter.	0	4176	4715	5400	5031	4401	4310	4756	5611	6899	8019
CPU	0.171	1.185	1.622	2.636	2.418	1.934	2.277	3.042	4.820	9.703	13.104
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.7741	0.3493	0.1802	0.1631	0.1586	0.1574	0.1569	0.1564	0.1564	0.1563
No. of non-zero R_i	0	1729	1806	1112	546	346	547	547	553	553	557

All the above computational experiments were conducted on a laptop with Intel® Core 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM. For the implementation of **CV** and **E_CV** models, the AMPL programming language and the CPLEX v.11 solver (with the default settings) were applied.

6. CONCLUSIONS

The portfolio approach presented in this paper has allowed the two popular in financial engineering percentile measures of risk, value-at-risk (VaR) and conditional value-at-risk ($CVaR$) to be applied. The computational experiments show that the proposed solution approach based on linear programming models provides the decision maker with a simple tool for evaluating the relationship between the expected and the worst-case portfolio return. The decision maker can assess the value of portfolio return and the risk level, and can decide how to invest in a real life situation comparing with the ideal (optimal) portfolio solutions. A risk-aversive decision maker wants to maximize the conditional value-at-risk. Since the amount by which losses in each scenario exceed VaR has been constrained of being positive, the presented models try to increase VaR and hence positively impact the objective functions. However, large increases in VaR may result in more historic portfolios (scenarios) with tail return, counterbalancing this effect. The concave efficient frontiers illustrate the trade-off between the conditional value-at-risk and the expected return of the portfolio. In all cases the CPU time increases when the confidence level decreases. The number of securities selected for the optimal portfolio for both the models varies between 1 and more than 50 assets. Those numbers show very little dependence on the confidence level α and the size of historical portfolio used as an input data. However, for the weighted-sum program, the weight parameter λ does have an influence on the number of stocks selected for the optimal portfolio.

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