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Competitive location under proportional choice: 1-suboptimal points on networks

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Abstract. This paper is concerned with a competitive or voting location problem on networks under a proportional choice rule that has previously been introduced by Bauer *et al.* (1993). We refine a discretization result of the authors by proving convexity and concavity properties of related expected payoff functions. Furthermore, we answer the long time open question whether 1-suboptimal points are always vertices by providing a counterexample on a tree network.

Keywords: competitive location, voting location, vertex optimality, discretization, centroid problem

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1. INTRODUCTION

Voting location problems are concerned with locating resources in some given space as the result of a collective election. They therefore incorporate the fact that the location sites themselves influence the utility drawn from the resources by the voters. Think, for example, of the problem of locating a school in a city and suppose that every individual wishes to have the facility as close as possible to its place of residence because the benefit enjoyed by the individual is a decreasing function of the distance traveled. The council of the city of Birmingham in the United Kingdom, for instance, wishes to find “placements that best meet the needs of children and young people and the aspirations of parents and carers, as close as possible to where they live” (Birmingham Council, 2011). We are faced with the problem of finding a compromise within the group of residents.

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Voting location problems are closely related to *competitive location models*, dating back to Hotelling (1929). Here we are concerned with the location of resources in space as well, but, rather than incorporating an explicit election process, location decisions are being made by independent decision-makers who will subsequently compete with each other, e.g. for market share when we think of locating facilities such as gas stations or supermarkets. *Sequential* locational competition is characterized by two types of players: leaders, who choose locations at given instants, anticipating the subsequent actions of later entrants, and followers, who make their location decisions based on the past decisions of the leaders. Even though the users in a voting location problem find compromises in order to locate resources that can be interpreted to be owned by a single decision maker, the voting process is such that a stable location is generally characterized by the nonexistence of a strong party of users who prefer alternative locations, that may be seen as possible locations of a competitive decision maker. Thus, although the focus of voting location problems is somewhat different, they may be interpreted as a special case of sequential competitive location problems. There is a strategic element that these problems share (Hansen *et al.*, 1990): independent decision makers influence the solution by making their decisions.

There are several kinds of compromises, i.e. rules for evaluating a voting process, that one may consider. Furthermore, in a mathematical model, one may consider different representations of the location space, e.g. networks or d -dimensional real spaces. A detailed overview of these and other classification criteria is given by Kress and Pesch (2012) and in the references therein. A well known concept for defining compromises is the *Condorcet point* (Hansen and Thisse, 1981). Here we are seeking point sets C of the location space such that no strict majority of voters prefers another point to any element of C . If the location space under consideration is represented by a network and the voters are located in the vertices of this network, it is well known that the existence of Condorcet points depends on the structure of the network and the distribution of the voters (see, for instance, Bandelt, 1985; Hansen and Thisse, 1981; Hansen *et al.*, 1986; Labbé, 1985). The potential nonexistence of Condorcet points in general networks led to the incorporation of a minimax objective in the literature, i.e. the search for points such that the maximal number of voters who prefer another point is minimal. An optimal solution to this optimization problem is called a *Simpson point* (Bandelt, 1985; Hansen and Labbé, 1988; Simpson, 1969).

The behavior of voters needs to be modeled by some kind of a choice rule in any voting location problem. Given a choice set, a choice rule is said to be binary, if it models a deterministic behavior of voters where each voter chooses exactly one (known) alternative. Under a proportional choice rule, the researcher can only derive probabilities of voting behavior. While the Condorcet and Simpson concepts make use of binary choice rules, Bauer *et al.* (1993) generalize these point sets to what they call *k-optimal* and *k-suboptimal* points ($k \in \mathbb{N} \cup \{\infty\}$ being a parameter for customer loyalty) by defining a proportional choice rule to model voting processes on networks (cf. also Hakimi, 1986, 1990, for discretization results on a follower location problem with multiple facilities under proportional choice). The paper at hand makes two contributions to the latter point sets. First, we refine a discretization property of Bauer *et al.* (1993) by proving convexity and concavity properties of related expected payoff

functions. Second, we answer the long time open question whether 1-suboptimal points are always vertices by providing a counterexample on a tree network. Additionally, we provide questions for future research.

The remainder of this paper is organized as follows. We first present graph theoretic fundamentals and some basic definitions in Section 2 in order to be able to define the proportional choice rule in Section 3. The main results of our research are presented in Section 4 and are complemented by examples and counterexamples in Section 5. The paper closes with a conclusion and remarks on future research in Section 6. Due to the close relationship of voting location and competitive location, we will use the terms user, customer and voter, facility and resource, as well as the terms number of voters and demand interchangeably throughout the paper.

2. NOTATION AND DEFINITIONS

In this section we will give some definitions and introduce the basic notation used throughout this paper. Some of the definitions are taken from Bauer *et al.* (1993) (cf. also Bandelt, 1985). We assume the reader to be familiar with the basic concepts of graph theory (see, for example, Gross and Yellen, 2004; Swamy and Thulasiraman, 1981).

We will denote a *network* by $N = (V, E, \lambda)$, with V ($|V| = n$) being the (finite) vertex set and E ($|E| = m$) being the (finite) edge set of the underlying graph. The mapping $\lambda : E \rightarrow \mathbb{R}^+$ defines the lengths of the network's edges. An edge $e \in E$ joining two vertices u and v is denoted by $e = [u, v]$. We assume that the networks considered in this paper are undirected, connected and that there are no multiple edges. Moreover, we assume that there are no loops at the vertices. We define the *points* x of a network N ($x \in N$) to be the elements of the edges (including all vertices) and denote the length of a shortest path (distance) connecting two points x and y of a network by $d(x, y)$. A *subedge* $[x, y]$ (or xy) of an edge $e \in E$ is determined by two points x and y on e ($x, y \in e$). The length of a subedge $[x, y]$ is denoted by $\lambda([x, y]) = \lambda(xy)$. A subedge defined by all points of an edge $[u, v] \in E$ without including the vertices u and v is denoted by (u, v) . We associate a (local) coordinate $x_{uv} \in [0, \lambda(uv)]$ with every edge $[u, v] \in E$ of a network N . Thus, we are able to define any point of the network. The direction of counting can be defined arbitrarily. The set of all points x on shortest paths between two vertices u and v of a network N is called the *interval* $I(u, v)$ between u and v , i.e. $I(u, v) = \{x \in N | d(u, v) = d(u, x) + d(x, v)\}$.¹ The interval is *ported* if for any pair of points $x \in I(u, v)$ and $y \notin I(u, v)$ every shortest path from x to y passes through u or v .

Given a network N , we will assume that there is a finite number of users located at the vertices of N . At each vertex there may be several users or none at all. Their number is described by a weight function $\pi : V \rightarrow \mathbb{R}_0^+$. We define $\pi(x) := 0$ for all $x \notin V$. π may not be equal to the zero function. For a subnetwork N' of N we denote by $\pi(V')$ the sum $\sum_{u \in V'} \pi(u)$ where V' is the vertex set of N' .

¹ Note that we denote open intervals of real space by (a, b) , $a, b \in \mathbb{R}$, $a < b$. Similarly, closed intervals of real space are denoted by $[a, b]$, $a, b \in \mathbb{R}$, $a \leq b$.

Let $N = (V, E, \lambda)$ be a finite network. Furthermore, let $[u, v] \in E$ be any edge and $i \in N$, $i \notin (u, v)$ be a point of the network (see Figure 1(a)).

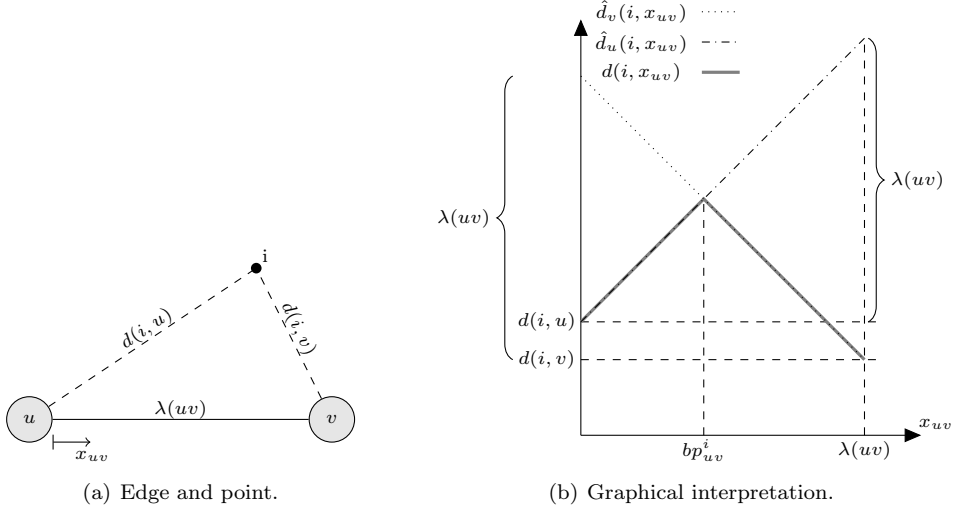


Fig. 1. Bottleneck points.

Define $\hat{d}_u(i, x_{uv})$ (and $\hat{d}_v(i, x_{uv})$) to be the length of the – not necessarily shortest – path from a point $x_{uv} \in [0, \lambda(uv)]$ to i via vertex u (or v) using the shortest path from u (or v) to i , that is $\hat{d}_u(i, x_{uv}) := d(i, u) + x_{uv}$ (and $\hat{d}_v(i, x_{uv}) := d(i, v) + \lambda(uv) - x_{uv}$). If there exists a point $x_{uv} = bp_{uv}^i \in (0, \lambda(uv))$ such that $\hat{d}_u(i, bp_{uv}^i) = \hat{d}_v(i, bp_{uv}^i)$, then $bp_{uv}^i = 0.5(d(i, v) + \lambda(uv) - d(i, u))$ (see Figure 1(b)). Such a point is generally referred to as a bottleneck point of the edge $[u, v]$ with respect to point i and defines the most remote point from i on edge $[u, v]$ (cf. also Hakimi, 1964; Hooker *et al.*, 1991). If $d(i, u) = d(i, v) + \lambda(uv)$ or $d(i, v) = d(i, u) + \lambda(uv)$, the bottleneck point is defined to be one of the vertices u or v , i.e.

$$bp_{uv}^i := \begin{cases} 0 & \text{if } d(i, u) = d(i, v) + \lambda(uv), \\ \lambda(uv) & \text{if } d(i, v) = d(i, u) + \lambda(uv). \end{cases} \quad (1)$$

We get

$$d(i, x_{uv}) = \begin{cases} d(i, u) + x_{uv} & \text{if } 0 \leq x_{uv} \leq bp_{uv}^i, \\ d(i, v) + \lambda(uv) - x_{uv} & \text{if } bp_{uv}^i < x_{uv} \leq \lambda(uv). \end{cases} \quad (2)$$

Now observe that, given an arbitrary edge $[u, v] \in E$ and any vertex $u_i \in V$ of the network, the function $d(u_i, x_{uv})$ is continuous in the variable x_{uv} on the interval $[0, \lambda(uv)]$. This is easily seen from (2), since an increase of a given x_{uv} by an arbitrarily small $\epsilon > 0$ (with $x_{uv} + \epsilon \leq \lambda(uv)$) can change the distance $d(u_i, x_{uv})$ by at most ϵ .

3. A PROPORTIONAL CHOICE RULE

Bauer *et al.* (1993) consider proportional choice in the context of a single facility voting location problem. The facility to be located in this problem corresponds to a leader's facility in the corresponding (two player) sequential competitive location problem. In order to make the distinction of leader and follower more intuitive, we will refine the notion and definitions of Bauer *et al.* (1993). This induces the need to revisit some of their propositions and theorems.

Let $N = (V, E, \lambda)$, $V = \{u_1, \dots, u_n\}$, be a (finite) network and let $x \in N$ be the leader's location and $y \in N$ the follower's location, respectively. We assume that the customers are homogenous in the sense that each of them demands exactly one unit of a (homogenous) commodity offered by the players. Then we define

$$E_L^k(x, y) := \begin{cases} \sum_{i=1}^n \pi(u_i) \cdot \frac{d(u_i, y)^k}{d(u_i, y)^k + d(u_i, x)^k} & \text{if } x \neq y, \\ \frac{1}{2} (\pi(V) + \pi(x)) & \text{if } x = y, \end{cases} \quad (3)$$

to be the expected number of customers visiting the leader's facility (index L) for any given $k \in \mathbb{N} \cup \{\infty\}$.² k may be interpreted as a parameter for customer loyalty, since a very large k models the deterministic (binary) behavior of the customers in the Condorcet model (cf. Bauer *et al.* (1993)). Similarly, the expected number of customers accommodating their demand at the follower's (index F) facility is defined to be

$$E_F^k(x, y) := \begin{cases} \sum_{i=1}^n \pi(u_i) \cdot \frac{d(u_i, x)^k}{d(u_i, x)^k + d(u_i, y)^k} & \text{if } x \neq y, \\ \frac{1}{2} (\pi(V) - \pi(x)) & \text{if } x = y. \end{cases} \quad (4)$$

It is easy to see that we have $E_L^k(x, y) + E_F^k(x, y) = \pi(N)$. Note that, if the follower's facility y is established at the same site as the leader's facility x , ties are broken such that all customers $v \in V \setminus \{x\}$ (are expected to) accommodate half of their demand at each of the facilities, while the customers located at x (are expected to) visit the leader's facility only. This assumption is designed to avoid trivial solutions (similar to Hakimi, 1990) and to guarantee continuity of the expected value functions (Proposition 3.1, cf. Bauer *et al.*, 1993). However, it can also be motivated from a practical point of view by interpreting it as enforcing a kind of local first mover advantage. That is, the leader may have superior information on the local market that can, for instance, be utilized for effective sales promotion.

Proposition 3.1.

Let N be a (finite) network. The expected values $E_L^k(x, y)$ and $E_F^k(x, y)$ are continuous functions in y on N . Furthermore, assuming $y \notin V$, they are continuous in x on N .

² Note that $E_q^k(x, y)$, $q \in \{L, F\}$, does not refer to the edge set E of the underlying network and that k is an index with respect to $E_q^k(x, y)$, while it is an exponent of the distances. Furthermore, note that we drop the index k when $k = 1$ in the remainder of the paper.

Proof. The proof is an immediate consequence of Propositions 1 and 2 of Bauer *et al.* (1993). ■

Observe that, given some $y \in V$ with $\pi(y) > 0$, $E_L^k(x, y)$ and $E_F^k(x, y)$ have exactly one discontinuity at $x = y$. It is easy to see that this is a result of the tie breaking rule given above.

In analogy to the Condorcet and Simpson concepts we define:

Definition 3.1 (Bauer *et al.* (1993)).

Let N be a (finite) network. Given two points $x, y \in N$, the value $E_F^k(x, y)/\pi(N)$ is the *relative k -rejection* of point x at point y . The *maximal relative k -rejection* of a point $x \in N$ is the value

$$\rho^k(x) := \max_{y \in N} \frac{E_F^k(x, y)}{\pi(N)}.$$

Definition 3.2 (Bauer *et al.* (1993)).

Let N be a (finite) network. A point $x \in N$ is called *k -optimal* if

$$E_L^k(x, y) \geq \frac{1}{2}\pi(N)$$

for all $y \in N$. A point $x \in N$ is called *k -suboptimal* if

$$\rho^k(x) = \min_{z \in N} \rho^k(z).$$

4. 1-SUBOPTIMAL POINTS

The following convexity property will prove useful in the sequel.

Proposition 4.1.

Let N be a (finite) network and $\hat{x} \in V$ be a fixed leader's location. The expected values $E_F(\hat{x}, x_{uv})$ are convex functions in the variable x_{uv} on $I = [0, \lambda(uv)]$ for all $[u, v] \in E$.

Proof. Given $\pi(u_i) \geq 0$ and Proposition 3.1, it is sufficient to show that the functions

$$e_{u_i, F}(\hat{x}, x_{uv}) := \frac{d(u_i, \hat{x})}{d(u_i, \hat{x}) + d(u_i, x_{uv})}$$

are convex in x_{uv} on the open interval $\bar{I} = (0, \lambda(uv))$ for all $\hat{x}, u_i \in V$ and $[u, v] \in E$. It is easy to see that the functions $e_{u_i, F}(\hat{x}, x_{uv})$ are convex (and non-increasing) in the (non-negative) distance $d(u_i, x_{uv})$ (a formal proof can be found in Hakimi, 1986). Now, given the previously derived continuity observation and the fact that $d(u_i, x_{uv})$ is the minimum of two linear functions in x_{uv} as shown in Figure 1(b), the assertion follows immediately. ■

Similarly we find:

Proposition 4.2.

Let N be a (finite) network and $\hat{y} \in V$ be a fixed follower’s location. The expected values $E_F(x_{uv}, \hat{y})$ are concave functions in the variable x_{uv} on $I = [0, \lambda(uv)]$ for all $[u, v] \in E$.

Proof. The proof is in analogy to the proof of Proposition 4.1. ■

As a consequence of Proposition 4.1 and Theorem 5 of Bauer *et al.* (1993) we may conclude:³

Proposition 4.3.

Let N be a network with vertex weight function $\pi : V \rightarrow \mathbb{R}_0^+$ and at least two positive vertex weights. Let $I(u, v)$ be a ported interval of N where $\pi(I(u, v)) = \pi(\{u, v\})$ and let y be any point of this interval where $u \neq y \neq v$. Then $E_F(x, u) > E_F(x, y)$ or $E_F(x, v) > E_F(x, y)$ for all $x \in N$.

Thus, the maximal relative 1-rejection of any point of the network is always reached at a vertex.

Definition 4.1.

Let N be a (finite) network and $k = 1$. We say R is the *upper envelope of rejection*, if $R(x_{uv}) = \max_{w \in V} E_F(x_{uv}, w)$ for any $x_{uv} \in [0, \lambda(uv)]$ and for all $[u, v] \in E$.

Given Proposition 4.3 and Proposition 4.2, it is easy to see that any 1-suboptimal point is at a vertex of the network N or at a point $x \in N$ where “rejection functions” that define the upper envelope of rejection intersect, i.e. where $E_F(x, w_1) = E_F(x, w_2) = R(x)$, $w_1, w_2 \in V$, $w_1 \neq w_2$. Making use of this discretization property, Bauer *et al.* (1993) design an algorithm of time complexity $\mathcal{O}(n^5)$ to determine all suboptimal points of a given network. It remained open if 1-suboptimal points are even always vertices. An example answers this long time open question for the case of tree networks. Consider the network of Figure 2 with all edge lengths equal to one and vertex weights $\pi(v) \in \{0, 1\}$ for all $v \in V$, the latter being indicated by numbers next to the vertices.

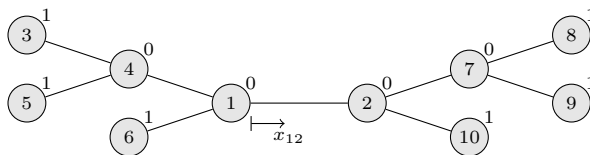


Fig. 2. Tree network.

Table 1 lists the relative rejection of vertex $i \in V$ (leader’s location) at vertex j (follower’s location) in cell (i, j) , when considering the example network. Thus, when restricting the set of potential locations to the vertex set, the maximal relative rejection is minimal at vertices 1 and 2. We denote this level of rejection by ρ_0 .

³ Note that, in their proof of Theorem 5 under the assumptions stated in Proposition 4.3, Bauer *et al.* (1993) essentially only show that $E_F(x, u) > E_F(x, y)$ or $E_F(x, v) > E_F(x, y)$ for all $x \in N \setminus \{u, v\}$ (not for all $x \in N$). Proposition 4.1 fixes this inaccuracy.

Table 1. Relative rejection of vertices.

	1	2	3	4	5	6	7	8	9	10
1	0.5	0.5	0.472	0.487	0.472	0.5095	0.486	0.462	0.462	0.486
2	0.5	0.5	0.462	0.487	0.462	0.486	0.487	0.472	0.472	0.5095
3	0.528	0.538	0.417	0.492	0.5	0.514	0.528	0.5	0.5	0.514
4	0.513	0.514	0.508	0.5	0.508	0.5	0.5	0.472	0.472	0.49
5	0.528	0.538	0.5	0.492	0.417	0.514	0.528	0.5	0.5	0.514
6	0.49	0.514	0.486	0.5	0.486	0.417	0.5095	0.486	0.486	0.5
7	0.514	0.513	0.472	0.5	0.472	0.49	0.5	0.508	0.508	0.5
8	0.538	0.528	0.5	0.528	0.5	0.514	0.492	0.417	0.5	0.514
9	0.538	0.528	0.5	0.528	0.5	0.514	0.492	0.5	0.417	0.514
10	0.514	0.49	0.486	0.5095	0.486	0.5	0.5	0.486	0.486	0.417

Figure 3 takes a closer look at edge $[1, 2]$ and corresponding rejection functions $E_F(x_{12}, w)$, $w \in V$. Note that, for the sake of clarity, only the subset of functions that define the upper envelope of rejection is depicted. We find that every point on edge $[1, 2]$ has a maximal relative rejection level smaller than or equal to ρ_0 . As a direct consequence we may conclude that there exists no 1-suboptimal point at a vertex of the network. Furthermore, when analyzing the remaining edges of the network, we find two points on edge $[1, 2]$, so_1 and so_2 , to be 1-suboptimal.

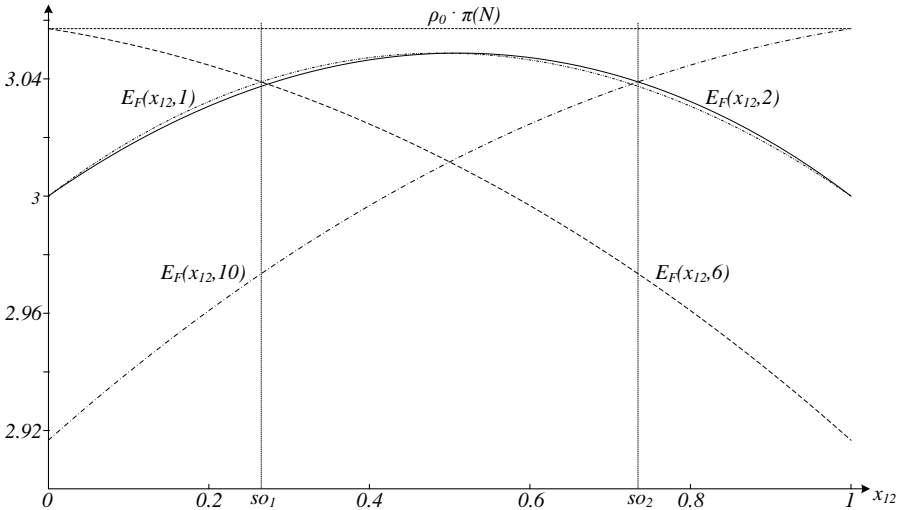


Fig. 3. Rejection functions and suboptimal points.

We close this section by providing a vertex optimality property for the family of connected networks with two vertices only.

Proposition 4.4.

Let $N = (V, E, \lambda)$, $V = \{u, v\}$, $E = \{\{u, v\}\}$ be a network with vertex weight function π and at least one positive vertex weight. Then $\rho^k(u) < \rho^k(x_{uv})$ or $\rho^k(v) < \rho^k(x_{uv})$ for all $x_{uv} \neq 0$ and $x_{uv} \neq \lambda(uv)$.

Proof. It is easy to see that $\rho^k(u) = \frac{\pi(v)}{\pi(N)}$ and $\rho^k(v) = \frac{\pi(u)}{\pi(N)}$. Therefore,

- if $\pi(u) = \pi(v)$, we have $\rho^k(u) = \rho^k(v) = 0.5$,
- if $\pi(u) > \pi(v)$, we have $\rho^k(u) < 0.5$,
- if $\pi(u) < \pi(v)$, we have $\rho^k(v) < 0.5$.

Thus, there always exists a k-optimal point, which, in combination with Theorem 6 of Bauer *et al.* (1993), i.e. the fact that k-optimal points are always vertices, proves the assertion. ■

5. EXAMPLES AND COUNTEREXAMPLES

When analyzing the example of Figure 2 in Section 4, the reader may have had several ideas on varying the network to achieve the vertex optimality property or simplifying the network without enforcing vertex optimality. This section aims at providing counterexamples on some of these potential ideas that may also raise questions for future research.

First, we show that the vertex optimality property may not hold even when $\pi(v) > 0$ for all $v \in V$. To this end, we augment the example network by introducing a sufficiently small constant $\epsilon > 0$ and adding this constant to all zero vertex weights (see Figure 4, where $\epsilon_1 = \epsilon_2 = \epsilon$). The relevant rejection functions for $\epsilon = 0.02$ are depicted in Figure 5(a) (all elements are in analogy to Figure 3). It is easy to see that the vertex optimality property does not hold. For future research, one may analyze how small the differences in vertex weights in general networks can get without enforcing vertex optimality.

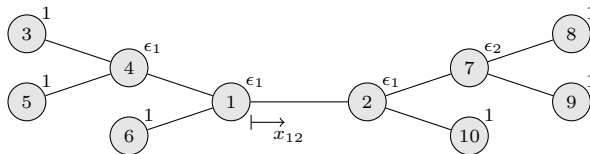
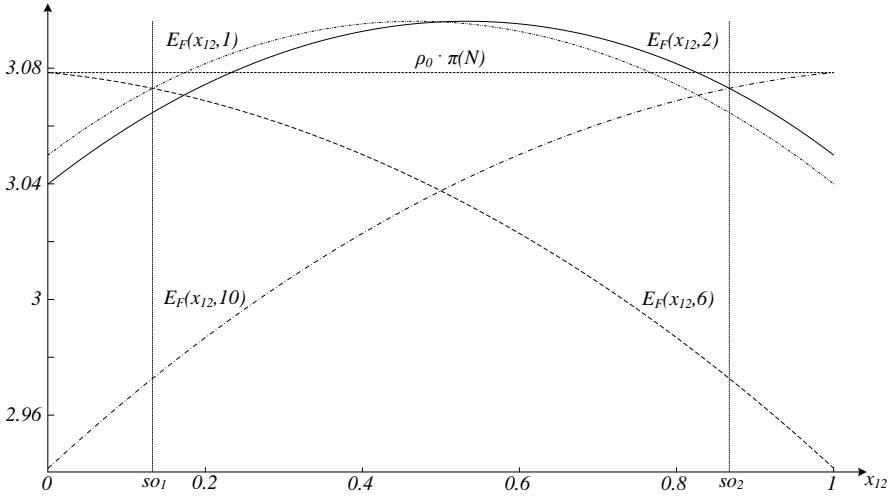
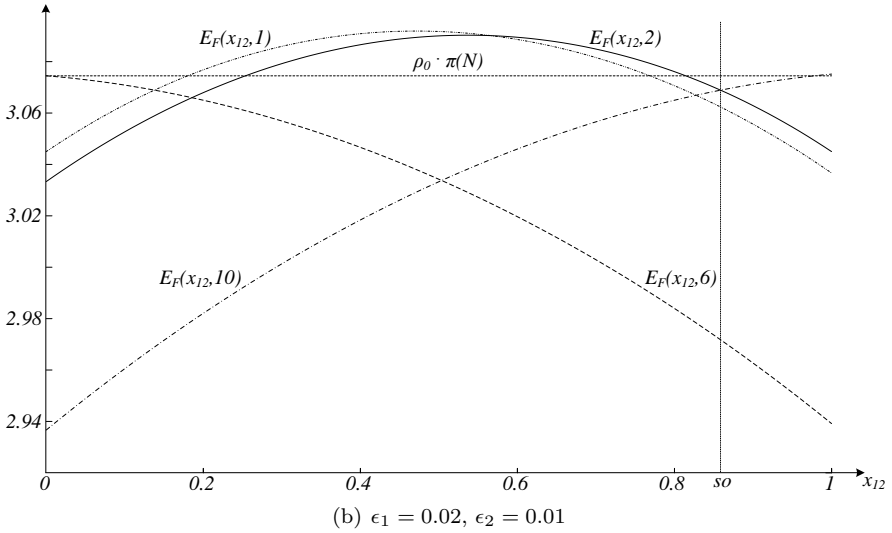


Fig. 4. No vertex weight equals zero.

We may conclude from the former example that there exist networks $N = (V, E, \lambda)$ that do not possess the vertex optimality property, even if $\pi : V \rightarrow \mathbb{N}$, i.e. $\pi(v) > 0$ and integer for all $v \in V$. To see this, suppose that all edge lengths and vertex weights of a network $N = (V, E, \lambda)$ are rational numbers. Represent those numbers as fractions of two integers and define c to be the least common multiple of their denominators. Then we can transform N into a network $N' = (V, E, \lambda')$ with



(a) $\epsilon_1 = \epsilon_2 = \epsilon = 0.02$.



(b) $\epsilon_1 = 0.02, \epsilon_2 = 0.01$

Fig. 5. Rejection functions and suboptimal points.

$\lambda'(uv) = c\lambda(uv)$ for all $[u, v] \in E$ and $\pi'(u) = c\pi(u)$ for all $u \in V$ without changing the ratio of the (expected) market shares of leader and follower for any feasible location setting.

Next, one may wonder if the symmetry of the example network plays a crucial role. Thus, we break the symmetry by setting $\epsilon_1 = 0.02 \neq \epsilon_2 = 0.01$ in the network of Figure 4. This results in the rejection functions shown in Figure 5(b). Again, the vertex optimality property does not hold.

We say that two neighbored vertices $v_1, v_2 \in V$ of a network $N = (V, E, \lambda)$ are being merged, if the corresponding edge length $\lambda(v_1, v_2)$ is reduced to zero, which is equivalent to introducing a new vertex v_{v_1, v_2} with $\pi(v_{v_1, v_2}) = \pi(v_1) + \pi(v_2)$ in the place of vertices v_1 and v_2 and edge $[v_1, v_2]$. Table 2 lists merging operations that the reader may have thought of in order to simplify the example network of Figure 2 without inducing vertex optimality. Observe that a 1-optimal point exists in all of the resulting networks, and thus any 1-suboptimal point is a vertex (cf. Theorem 6 of Bauer *et al.* (1993)).

Table 2. *Merging vertices.*

merge vertices	3,5	3,5 and 8,9	3,4	3,4 and 7,9
1-optimal points	$v_{3,5}$	$v_{3,5}$ and $v_{8,9}$	$v_{3,4}$	$v_{3,4}$ and $v_{7,9}$
merge vertices	6,1	6,1 and 2,10	3,5 and 7,9	3,5 and 2,10
1-optimal points	$v_{6,1}$	$v_{6,1}$ and $v_{2,10}$	$v_{3,5}$ and $v_{7,9}$	$v_{3,5}$
merge vertices	3,4 and 2,10	3,5 and 6,1	3,5; 6,1; 2,10 and 8,9	
1-optimal points	$v_{2,10}$	$v_{3,5}$	$v_{6,1}$ and $v_{2,10}$	

In analogy to Table 2, Table 3 refers to deleting vertices and their incident edges in the example network. Again, a 1-optimal point exists in all of the resulting networks.

Table 3. *Deleting vertices.*

delete vertices	3	6	3,5	3,6	3,8	3,10	6,10	3,5,8
1-optimal points	10	10	8,9	8,9	6,10	8,9	3,5,8,9	10
delete vertices	3,5,10	3,6,8	3,6,10	3,5,8,9	3,5,8,10	3,6,8,10		
1-optimal points	8,9	10	8,9	6,10	6,9	5,9		

6. CONCLUSION AND FUTURE RESEARCH

This paper has been concerned with voting location problems under a proportional choice rule, introduced by Bauer *et al.* (1993). We have refined a known discretization property in Proposition 4.3 by proving convexity and concavity properties of related expected payoff functions in Propositions 4.1 and 4.2. Moreover, we have answered the long time open question whether 1-suboptimal points are always vertices by providing a counterexample on a tree network in Section 4. Additionally, a discretization property for the family of connected networks with two vertices only has been derived in Proposition 4.4. This property raises an interesting question for future research: Does the vertex optimality property carry over to the more complex family of chain networks $N = (V, E, \lambda)$ with vertex set $V = \{u_1, \dots, u_n\}$ and edge set $E = \{[u_i, u_{i+1}] | i = 1, \dots, n - 1\}$ for $k = 1$?

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