Allocating Pooled Inventory According to Contributions and Entitlements

Yigal Gerchak*

Abstract. Inventory pooling, whether by centralization of stock or by mutual assistance, is known to be beneficial when demands are uncertain. But when the retailers are independent, the question is how to divide the benefits of pooling. e consider a decentralized inventory-pooling scheme where a retailer's entitlements to the allocation during a shortage depend on his/her contributions to the pool. We derive the Nash equilibrium and specialize it to symmetric cases.

1. INTRODUCTION

Inventory pooling in a decentralized system raises the issue of how to divide benefits among the parties, which is typically done using concepts from cooperative games (e.g., Gerchak and Gupta 1991; Hartman *et al.*, 2000; Müller *et al.*, 2002; Montruc-chio *et al.*, 2012); In particular, if the before-pooling optimal expected profits are $\pi_1^*(q_1^*)$ and $\pi_2^*(q_2^*)$, and the optimal expected profit after pooling is $\pi^*(q^*)$, the gain, $\pi^*(q^*) - \pi_1^*(q_1^*) - \pi_2^*(q_2^*)$ could be divided proportionally to $\pi_1^*(q_1^*)$ and $\pi_2^*(q_2^*)$. That allocation can be shown to belong to the (non-empty) core. We propose a very different "operational" *non-cooperative* scheme according to which the parties have entitlements that are increasing functions of their *contributions* to the pool rather than a typical division of benefits. The idea was proposed by Ben-Zvi and Gerchak (2012, sec. 11.5); but here, we analyze the consequences of a somewhat-modified scheme. The scheme can be shown to be beneficial to all parties vis-á-vis a no-pooling situation, namely is in core.

The scheme works as follows: each party (e.g., retailer) contributes a quantity of its choice to the pool. There is no inventory at the retailers' locales. The choice of these quantities is simultaneous. If the quantity of pooled inventory is not sufficient to

* Tel Aviv University, Department of Industrial Engineering, Israel, e-mail: yigal@post.tau.ac.il

meet all realized demands, a retailer whose contribution exceeds its demand receives an allocation equal to its demand. The other retailer (whose demand exceeds its contribution) receives the rest of the inventory. If both demands exceed the respective contributions, each retailer receives its contribution. The scheme can be shown to be beneficial for all parties vis-ŕ-vis a no-pooling scenario. Mathematically (but not in type of motivations), our model is related to Shao *et al.* (2011).

The simultaneous selection of quantities to contribute to the pool is a non-cooperative game. In a symmetric scenario (unit cost, revenue, demand distributions), we solve directly for the (common) order quantity. We provide an example.

2. GENERAL MODEL

Let r_i be the unit revenue and c_i unit production cost of party i, $r_i > c_i \forall i$. The distributions of the independent demands, X_1 and X_2, are F_1 and F_2. The parties are simultaneously looking for their best contribution quantities q_1 and q_2. The actual allocations to the parties as a function of demand realizations (x_1, x_2) are denoted by a_1 and a_2.

The policy (contract) is as follows.

If $x_i \leq q_i$, then $a_i = x_i$;

if, in addition, $x_j \leq q_j$ then $a_j = x_j$;

if $x_j > q_j$ and $x_i + x_j \leq q_i + q_j$, then $a_j = x_j$;

if $x_j > q_j$ and $x_i + x_j > q_i + q_j$, then $a_j = q_i + q_j - x_i (\leq x_j)$. [This is the case where inventory pooling helps party j].

If $x_i > q_i$ and $x_j > q_j$, then $a_i = q_i$ and $a_j = q_j$. ‖

So party $i's$ expected profit is:

$$
\begin{aligned}
E_i = -c_i q_i + r_i \Bigg\{ & \int_{x_i=0}^{q_i} x_i \int_{x_j=0}^{q_j} f_i(x_i) f_j(x_j) \, dx_j dx_i \\
& + \int_{x_i=q_i}^{q_i+q_j} \int_{x_j=0}^{q_i+q_j-x_i} x_i f_i(x_i) f_j(x_j) \, dx_j dx_i \\
& + \int_{x_i=q_i}^{q_i+q_j} \int_{x_j=q_i+q_j-x_i}^{q_j} (q_i + q_j - x_j) f_j(x_j) f_i(x_i) \, dx_j dx_i \\
& + \int_{x_i=q_i+q_j}^{\infty} \int_{x_j=0}^{q_j} (q_i + q_j - x_j) f_j(x_j) f_i(x_i) \, dx_j dx_i \\
& + \int_{x_i=q_i}^{\infty} \int_{x_j=q_j}^{\infty} q_i f_i(x_i) f_j(x_j) \, dx_j dx_i \Bigg\}
\end{aligned}
$$

so:

$$dE_i/dq_i = -c_i + r_i \bigg\{ 2\overline{F}_j(q_j)\overline{F}_i(q_i) - q_i f_i(q_i)\overline{F}_j(q_j)$$

$$+ q_j F_j(q_j) - q_i f_j(q_j) F_i(q_i + q_j)$$

$$+ \int\limits_{x_i=q_j}^{q_i+q_j} F_j(q_i + q_j - x_i)\,dx_i + F_j(q_j)\overline{F}_i(q_i + q_j) - q_i f_j(q_j) \bigg\}$$

Note that E_i is an explicit function of only c_i and r_i. On the other hand, it depends on both F_i and F_j.

In the symmetric case ($c_1 = c_2 \equiv c$, $r_1 = r_2 \equiv r$), $F_1 = F_2 \equiv F$, where we will look for a symmetric equilibrium, $q_1 = q_2 \equiv q$, we have ($r > c$)

$$dE/dq = -c + r \bigg\{ 2\left[\overline{F}(q)\right]^2 - qf(q)\overline{F}(q) + qF(q) - qf(q)F(2q)$$

$$+ F(q)\overline{F}(2q) - qf(q) + \int\limits_{q}^{2q} F(2q - x)\,dx \bigg\}$$

$$dE/dq|_{q=0} = -c + 2r > 0 \Rightarrow q^* > 0$$

3.   UNIFORMLY DISTRIBUTED DEMANDS

Here $X \sim U[0,1]$, so $0 \leq q \leq 1$.

If $q \leq \frac{1}{2}$, the optimality condition becomes:

$$-c + r\left(-3q^2 - 3q + 2\right) = 0$$

i.e., $3q^2 + 6q - 4 + \frac{2c}{r} = 0$.

So:

$$q^* = 1 - \frac{\sqrt{21 - \frac{6c}{r}}}{3} \quad \left(< \frac{1}{2}\right)$$

.

If $q > \frac{1}{2}$, the optimality condition becomes:

$$-c + r\left(\frac{9}{2}q^2 - 7q + 2\right) = 0$$

i.e., $q^2 - 14q + 4 - \frac{2c}{r} = 0$

$$\Rightarrow q^{**} = 7 - \sqrt{45 + \frac{2c}{r}}$$

However, $q^{**} < \frac{1}{2}$. Thus, the solution in this range is either boundary $q = \frac{1}{2}$ or $q = 1$.

4.   CONCLUDING REMARKS

In partnerships, the partners' entitlements to profits are often based on their ownership shares. We use a similar philosophy in allocating scarce inventories. That translates to a non-cooperative game.

Issues that were not explored in this context but might be of interest include:

1) Non-Linear productions costs (Gerchak and Schwarz, 2014).
2) Holding some inventory at the retailers' locales ("partial pooling"), possibly with transshipment costs from the pool.
3) More insight into the case of non-symmetric parameters.

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