Computer Science • 15 (3) 2014 http://dx.doi.org/10.7494/csci.2014.15.3.329

Jan Kucwaj

AN ALGORITHM OF SEMI-DELAUNAY TRIANGULATION OF POINTS CLOUD SCATTERED ON A SURFACE

Abstract *The purpose of this paper is to generalize the Delaunay[13] triangulation onto surfaces. A formal definition and an appropriate algorithm are presented. Starting from a plane domain Delaunay triangulation definition, a theoretical approach is evolved (which is a background for further considerations). It has been proven that, in the case of a plane surface, the introduced Delaunay triangulation of surfaces is identical to classical Delaunay triangulation of the plane domain. The proposed algorithm is implemented. and numerical results are shown.*

Keywords Delaunay triangulation, surface meshing, surface reconstruction, advancing front technique.

1. Introduction

The purpose of this paper is to generalize the Delaunay [1, 2, 5] triangulation onto surfaces. A formal definition and an appropriate algorithm are presented. Starting from a plane domain Delaunay triangulation definition, a theoretical approach is evaluated (which is a background for further considerations). It has been proven that, in the case of a plane surface, the introduced Delaunay triangulation of surfaces is identical to classical Delaunay triangulation of the plane domain. The main idea of the algorithm is a coupling of the Delaunay property and the the advancing front technique [8, 9]. For the sake of implementation, it is assumed that points are given on a multi-connected surface with a finite number of internal loops.

It is obviously important to perform an optimal triangulation (composition) process of these points; i.e., their mapping onto the surface. It is worth noting that surfaces can be represented by B-spline functions or NURBS approximations [7] as well. For finite elements or finite differences applications [9, 11], it is necessary to generate meshes with a prescribed density. On the other hand, triangulation of a given surface is a two-stage process: firstly, a set of points is generated, then they are appropriately connected to form triangles.

The main purpose of thi paper is to present a new algorithm for the triangulation process of a surface spanning a set of points, [5]. The proposed algorithm is based on the Advancing Front Techniques (AFT) coupled with the Delaunay triangulation, [8]. The latter method dealt only with triangles in 2D; i.e., a plane. But the proposed approach uses triangles for mapping a surface; e.g., spherical. The algorithm is to be coded, and numerical results are to be generated.

2. Preliminaries

Details about triangulation of a plane with openings is described in the current section. It is assumed that the boundary of the domain consists of k components. In Figure 1, the domain is the area between the five outer straight lines and three inner lines. The area inside the three inner lines is emptiness. Hence, the domain in Figure 1 consists of two components.

It is assumed that we have a $k-$connected polygon Ω with k boundaries of n_i vertices $i = 1, \dots, k$, so the set of boundary points can be denoted:

$$\Gamma = \{\mathbf{P_1}, \dots, \mathbf{P_{n_1}}, \dots, \mathbf{P_{n_i}}, \dots, \mathbf{P_{n_{i+1}}} \dots, \mathbf{P_{n_k}}\} \tag{1}$$

We define the following sets of straight line segments:

$$S_1 = \{\mathbf{P_1P_2}, \mathbf{P_2P_3}, \dots, \mathbf{P_{n_1}P_1}\} S_i = \{\mathbf{P_{n_{i-1}+1}P_{n_{i-1}+2}}, \dots, \mathbf{P_{n_i-1}P_{n_{i-1}+1}}\}, \quad \text{for} \tag{2}$$

It is possible to write the boundary of Ω in the following form:

$$\partial\Omega = \bigcup_{i=1}^{k} S_i \tag{3}$$

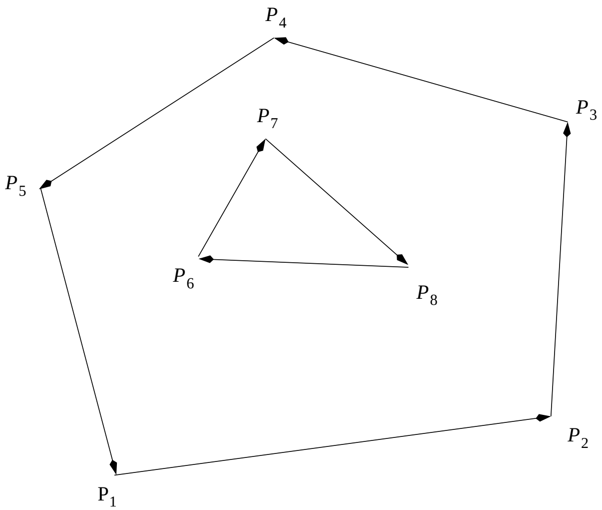

Figure 1. Boundary orientation with respect to the domain.

where S_1 is the external component of the boundary, and $S_2, \dots, S_k$ are the internal components (loops) of $\partial\Omega$ like in Figure 1. We assume that curve orientation is defined by increasing-points numbering and concurs with the counter-clockwise orientation of the boundary with respect to the domain.

Let Θ be a set of internal points of the domain Ω.

Definition 1 *A set of triangles $\mathcal{T} = \{T_i\}_{i=1}^{N_T}$ is called the triangulation of plane polygonal shaped domain Ω on sets Θ and Ω if the following conditions are satisfied:*

(i) $\overline{\Omega} = \bigcup_{i=1}^{N_T} \overline{T}_i$,

(ii) $\forall\ i, j = 1, 2, \dots, N_T,\ i\ \neq j$ *is:* $T_i \cap T_j = \emptyset$,

(iii) *all the triangles vertices of $\mathcal{T}$ come from the set $\Gamma \cup \Theta$*

(iv) $\forall \mathbf{P} \in \Gamma \cup \Theta\ \exists\ k\ 1 \leqslant k \leqslant N_T$, *such that* $\mathbf{P}$ *is a vertex of the* T_k.

In the case of the convex polygon $\Omega \subset \mathbb{R}^2$, the definition 1 is equivalent to the definition in [13] on set $\Gamma \cup \Theta$.

Definition 2 *A triangulation $\mathcal{T}$ of a polygonal set $\Omega \subset \mathbb{R}^2$ is the Delaunay triangulation if for arbitrary $T \in \mathcal{T}$ the circumscribed circle on it does not contain in its interior any point from $\Gamma \bigcup \Theta$, which could create a triangle with any pair of vertices of the T contained in Ω.*

3. The Delaunay triangulation of set of points in 2-D domain

The presented algorithm of 2-D domain triangulation is based on the following papers: [4, 8, 9]. It is based on the Advancing Front Technique (AFT) and satisfies the Delaunay condition. It is assumed that the set of boundary segments S (see 2) is

a starting front. During the algorithm realization, the front will vary. The front will eventually shrink to zero.
$\mathcal{S}'_{\mathcal{D}}$ — Delaunay part of the front,
$\mathcal{S}'_{\mathcal{N}-\mathcal{D}}$ — non-Delaunay part of the front.
At the algorithm beginning:

$$\mathcal{S}^0_D = \emptyset \quad \text{and} \quad \mathcal{S}^0_{N-D} = \mathcal{S} \tag{4}$$

In the successive steps of the algorithm, S^i_D and S^i_{N-D} are defined by induction. During the algorithm, both parts of the front are being changed – but the following condition is still being kept:

$$\mathcal{S}^i = \mathcal{S}^i_D \cup \mathcal{S}^i_{N-D} \quad \text{and} \quad \mathcal{S}^i_D \cap \mathcal{S}^i_{N-D} = \emptyset \quad \text{for} \quad i = 1, \dots n, \tag{5}$$

where i is the number of the step in AFT.

Let's define inductively S^i_D and S^i_{N-D} for which the start conditions are defined in 4. If new sides **PA**, **BP** of the triangle **ABP** (Fig. 4a) are not on the front, then they are added to S^i_D, provided that there is one element lying on the circle passing through the points **A**, **B**, **P** besides **A**, **B**; otherwise, they are added to S^i_{N-D}. It results in obtaining successive fronts S^{i+1}_D and S^i_{N-D}. The segment **AB** is always removed from the front. In this way, condition 5 is still satisfied.

The successive step of the AFT starts at choosing from front $\mathcal{S}^i$ a straight line segment **AB** (it is recommended that the smallest one is chosen). Let Σ^i denote the set of front boundary points and internal points of the domain bounded by the front. At the beginning $\Sigma^i = \Theta \bigcup \Gamma$ (definition 1). For a fixed $r > 0$ (usually r is equal to multiplicity of the segment length), the set of candidates associated with **AB** is defined in the following way:

Definition 3 $C^r_{\mathbf{AB}} = \{\mathbf{P} \in \Sigma^i$*: where the vectors* **AB** *and* **BP** *create a pair positively oriented and* ***P*** *belongs to the circle with radius* r *and passing throught the points* ***A***, ***B***$\}$

The relation $\preceq$ on the set $C^r_{\mathbf{AB}}$:

$$\forall\, \mathbf{P},\, \mathbf{Q} \in C^r_{\mathbf{AB}}\ \mathbf{P} \preceq \mathbf{Q} \Longleftrightarrow \quad \text{if} \quad \mathbf{P} \in \overline{K}_{\mathbf{ABQ}} \tag{6}$$

where $K_{\mathbf{ABQ}}$ is a circle passing throughout **A**, **B**, **Q** and $\overline{K}_{\mathbf{ABQ}}$ is its closure is introduced. The relation $\preceq$ is reflexive, transitive and total (Fig. 2a, b)

Relation $\preceq$ is not an ordering relation, because it is not antisymmetric. This means it does not satisfy the following condition:

$$\forall\, \mathbf{P}, \mathbf{Q} \in C^r_{\mathbf{AB}}\ [\mathbf{P} \preceq \mathbf{Q} \wedge \mathbf{Q} \preceq \mathbf{P}] \Longrightarrow \mathbf{P} = \mathbf{Q} \tag{7}$$

It means that the minimal element in the set $C^r_{\mathbf{AB}}$ is not uniquely determined. To ensure uniqueness in the set $C^r_{\mathbf{AB}}$, the following relation is also introduced:

$$\forall\, \mathbf{P},\, \mathbf{Q} \in \mathbf{C}^r_{\mathbf{AB}}\ \mathbf{P} \cong \mathbf{Q} \Longleftrightarrow \text{if}\, [\mathbf{P} \in \overline{K}_{\mathbf{ABQ}} \wedge \mathbf{Q} \in \overline{K}_{\mathbf{ABP}}] \tag{8}$$

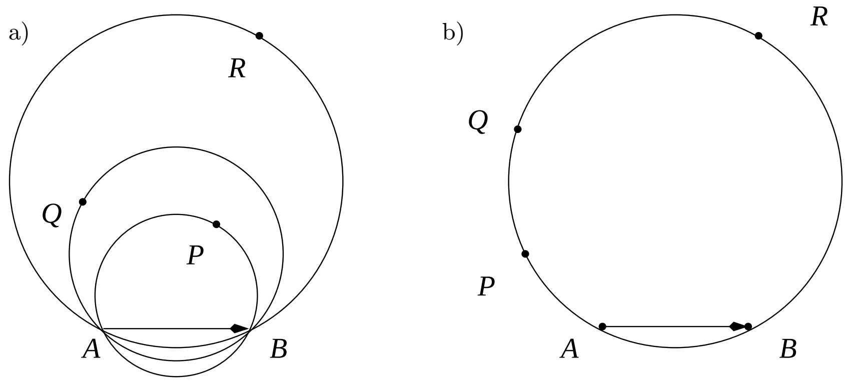

Figure 2. a) Relation $\preceq$; b) Singular case.

The relation $\cong$ is an equivalence relation (Fig. 2b). In the quotient set, the next relation is defined; namely:

$$\forall\, \mathbf{P}, \mathbf{Q} \in \mathbf{C}^r_{\mathbf{AB}}\ [\mathbf{P}]_{\cong} \preceq_{\cong} [\mathbf{Q}]_{\cong} \iff \mathbf{P} \preceq \mathbf{Q} \tag{9}$$

It is easy to check that $\preceq_{\cong}$ does not depend on the choice of class representative, it means it is right defined as well as reflexive, antisymmetric, total, and transitive (i.e., it is an ordering relation).

Reflexivity, totality, and transitivity are consequences of these properties for $\preceq$. To show antisymmetricity, we assume that $[\mathbf{P}] \preceq_{\cong} [\mathbf{Q}]$ i $[\mathbf{Q}] \preceq_{\cong} [\mathbf{P}]$, which means that $\mathbf{P} \in \overline{K}_{\mathbf{ABQ}}$ and $\mathbf{Q} \in \overline{K}_{\mathbf{ABP}}$, what leads to identity of circles $K_{\mathbf{ABQ}}$ and $K_{\mathbf{ABP}}$ or $\mathbf{P}$, $\mathbf{Q}$ are in the relation $\cong$ and in the consequence $[\mathbf{P}]_{\cong} = [\mathbf{Q}]_{\cong}$. $\square$

Remark 1 *Equivalence class of the point* $\mathbf{P} \in S^r_{\mathbf{AB}}$ *is equal to:* $[\mathbf{P}]_{\cong} = \{\mathbf{Q} \in S^r_{\mathbf{AB}} :$ $\mathbf{Q}$ *lies on the circumference passing throughout the points* $\mathbf{A}, \mathbf{B}, \mathbf{P}\}$.

Equivalence class may contain more than one element; such a situation is called a singular case (Fig. 2b).

For the sake of the efficient evaluation of the relation $\cong$, the local coordinate system is introduced (Fig. 3a) with the basis $\mathbf{u} = \frac{\mathbf{AB}}{||\mathbf{AB}||}$ of unit versor $\mathbf{v}$ perpendicular to $\mathbf{u}$ and being with it a positively defined pair. The point:

$$\mathbf{O} = \frac{1}{2}(\mathbf{A} + \mathbf{B}) \tag{10}$$

is taken as the origin of the coordinate system.

In the coordinate system, the center $\mathbf{C}_1$ of circle $K_{\mathbf{ABP}}$ (Fig. 3a) has the coordinate $u = 0$ and v appointed from condition $||\mathbf{AC_1}|| = ||\mathbf{PC_1}||$. It turns out that the definition of the relation in the set $C^r_{\mathbf{AB}}$ agrees with the order of coordinates $\mathbf{v}$ of the appropriate circles, namely the following theorem is true:

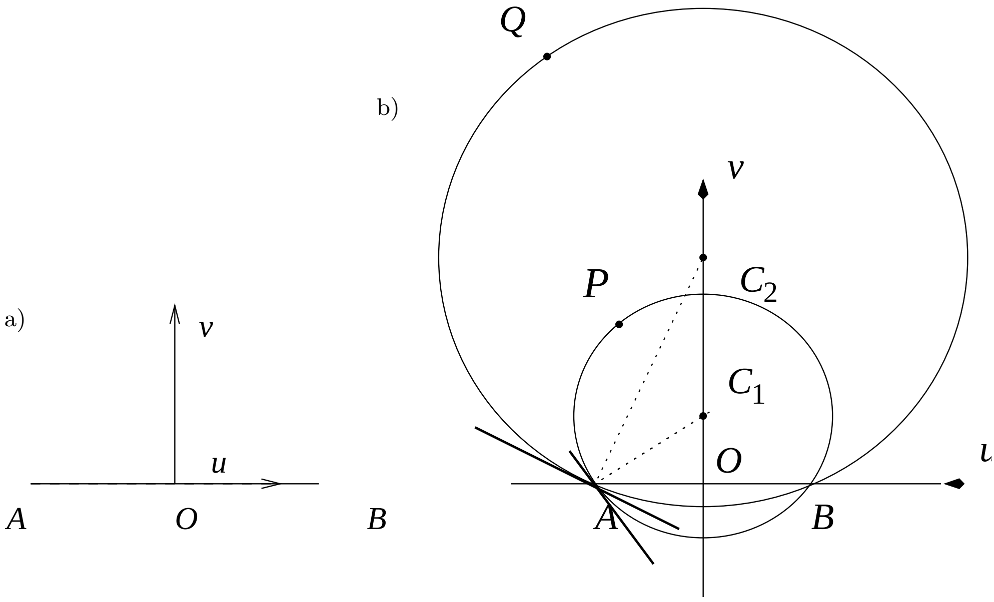

Figure 3. a) local coordinate system; b) Relation $\preceq$ and v coordinate.

Theorem 1 *Let $\mathbf{P}, \mathbf{Q} \in S^r_{\mathbf{AB}}$, then the following identity is true:*

$$\mathbf{P} \preceq \mathbf{Q} \Longleftrightarrow \textit{center coordinate of } K_{\mathbf{ABP}} \\ \textit{is less than coordinate } v \textit{ of the center} \quad K_{\mathbf{ABQ}} \qquad (11)$$

Proof: the centers of circles $K_{\mathbf{ABP}}$, $K_{\mathbf{ABQ}}$ lie on the v axis and are the intersection points straight lines appropriately perpendicular to the tangents at point $\mathbf{A}$ to these circles. If $\mathbf{P} \preceq \mathbf{Q}$, then the mutual placement of $K_{\mathbf{ABP}}$ and $K_{\mathbf{ABQ}}$ is as in figure 3b. The intersection of the straight line perpendicular to the tangents at point $\mathbf{A}$ to the $K_{\mathbf{ABP}}$ with axis v is beneath the analogous intersection for circle $K_{\mathbf{ABQ}}$. $\square$

The theorem 1 efficiently allows us to check the $\preceq$ relation. The relation $\preceq_{\cong}$ is an ordering relation in finite quotient set $C^r_{\mathbf{AB}}/\cong$, and a minimal uniquely-defined element exists with respect to this relation. The element $[\mathbf{P}_{\min}]_{\cong}$ is an abstract class, so it may contain more than one element, which means that it is the singular case (Fig. 4a). From this class, one element is chosen which has the greatest of the minimal angles of the triangles formed by segment $\mathbf{AB}$ and the points of the class. It is easy to see that the considered criterion is equivalent to the choice of a point $\mathbf{P}$, whose v coordinate in the local coordinate system is maximal (Fig. 4).

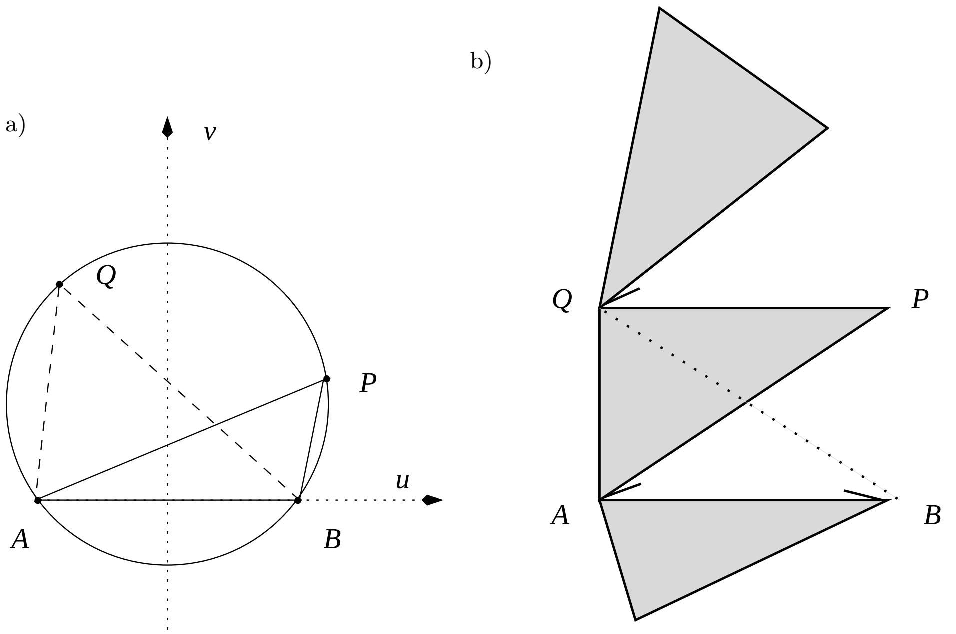

Figure 4. a) Singular case b) The intersection with non-Delaunay front part S^i.

After the point $\mathbf{P}$ selection, it is necessary to check the following conditions:

$$\mathbf{PA} \cap S^i_{N-D} \in \{\emptyset, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{P}\}, \mathbf{PA}\} \tag{12}$$

$$\mathbf{PB} \cap S^i_{N-D} \in \{\emptyset, \{\mathbf{B}\}, \{\mathbf{B}, \mathbf{P}\}, \mathbf{PB}\} \tag{13}$$

The conditions 12, 13 ensure that the newly-created **ABP** triangle does not overlap the existing triangles (Fig. 4b – marked triangles). In formulas 12, 13 appears a non-Delanay part of the front, because, with respect to the Delaunay condition, it is necessary to check the condition only for this part of the front.

The front modification describes the following algorithm:

Algorithm 1 *1. Remove segment* $\mathbf{AB}$ *from the front* $S^{i+1} = S^i \setminus \{\mathbf{AB}\}$

2. IF($\mathbf{PA} \in S^i_D$*) THEN*
$S^{i+1}_D = S^i_D \setminus \{\mathbf{PA}\}$
ELSE
$S^{i+1}_D = S^i_D \bigcup \{\mathbf{PA}\}$
ENDIF

3. IF($\mathbf{BP} \in S^i_{N-D}$*)THEN*
$S^{i+1}_{N-D} = S^i_{N-D} \setminus \{\mathbf{BP}\}$
ELSE
$S^{i+1}_{N-D} = S^i_{N-D} \bigcup \{\mathbf{BP}\}$
ENDIF.

Thus, the algorithm of the Delaunay triangulation can be presented according to the following scheme:

Algorithm 2

1. $S^0_{N-D} = S$
2. $S^0_D = \emptyset$

   Perform points from a) to ...e) for $i = 0, 1, 2, \ldots$ while $S^i \neq \emptyset$:

   (a) *IF($S^i_{N-D} \neq \emptyset$)THEN*
   find the last front segment $\mathbf{AB} \in S^i_{N-D}$
   ELSE
   IF($S^i_D \neq \emptyset$) THEN
   find the last segment of the S^i_D
   ELSE
   finish the triangulation
   ENDIF.

   (b) *appoint the set of candidates $C^r_{\mathbf{AB}}$ dla $r = \mathbf{AB}$ according to the definition 3*

   (c) *Find $[\mathbf{P}^*]_{\cong}$ as the minimal element in the set $C^r_{\mathbf{AB}}/\cong$*

   (d) *IF($\mathbf{P}^*$ satisfies the conditions 12,13)THEN*
   add the triangle to the list of the triangles and modify the front S^{i+1}_D i S^{i+1}_{N-D} by using the algorithm of the front modification
   ELSE

   (e) $C^r_{\mathbf{AB}} \longleftarrow C^r_{\mathbf{AB}} \setminus \{\mathbf{P}^*\}$
   IF($C^r_{\mathbf{AB}} \neq \emptyset$) THEN
   go to 2d
   ELSE
   $r \longleftarrow r + D$, where D is the front length
   go to 2b
   ENDIF ENDIF.

It will be proven that this algorithm leads to the Delaunay triangulation.

Theorem 2 *(Delaunay theorem [13]) For a triangulation $\mathcal{T} = \{T_i\}_{i=1}^{N_\mathcal{T}}$ – of a finite set of cloud of points in the plane [13], the following is true: $\forall\ T_i, T_j \in \mathcal{T}\ K_{T_i}$ does not contain vertices of T_j in its interior and vice-versa, if and only if $\forall\ T_i, T_j \in \mathcal{T}$ sharing a common edge T_i does not contain vertices of T_j in its interior and vice versa.*

The right-hand side of the equivalence of the theorem 3 allows us to define equivalently to the definition 2, the definition of the Delaunay triangulation of a 2-D domain; namely: The theorem 2 can be generalized for triangulation of a polygonal domain:

Theorem 3 *For any triangulation of $2-D$ polygonal-shaped domain, the following equivalence is true: $\forall\ T_i, T_j \in \mathcal{T}\ K_{T_i}$ does not contain vertices of T_j in its interior and vice-versa if and only $\forall\ T_i, T_j \in \mathcal{T}\ K_{T_i}$ sharing a common edge does not contain* *vertices of T_j in its interior and vice-versa. (in this case, it is said about mutual Delaunay placement of the both triangles).*

Proof: if the condition of the left hand side of the equivalence is satisfied, then $\forall\ T_i, T_j \in \mathcal{T}\ K_{T_i}$ sharing the common edge K_{T_i} does not contain vertices of T_j in its interior and vice-versa, because the vertices of K_{T_i} create a right triangle of the triangulation with vertices of T_j. On the contrary, if the condition of the right hand side of the equivalence is satisfied, then from the theorem 2 follows that the conditions of the definition 2 are satisfied. □

It turns out that the algorithm 2e leads to the Delaunay triangulation of the polygonal-shaped domain. Before the formulation of the main theorem, the following lemma will be proven:

Lemma 1 *Triangulation $\mathcal{T}$ of an arbitrary polygonal domain Ω is the Delaunay triangulation if and only $\forall\ T_i, T_j \in \mathcal{T}\ K_{T_i}$ sharing a common edge K_{T_i} does not contain vertices of T_j in its interior* ***or*** *vice-versa.*

Proof: the conclusion of the right hand side from the left side of the equivalence follows from theorem 3, as the conjunction implies disjunction. On the other side, if for any two adjacent triangles $\triangle ABC$ and $\triangle ABD$ (Fig. 2) point **D** does not belong to the circle circumscribed about the triangle $\triangle ABC$ and point **C** belongs to the circle circumscribed about the triangle $\triangle ABD$, then point **C** would be in the lower part of the circle segment $\triangle ABC$ defined by the **AB** straight line segment, which would mean that point **D** belongs to the circle circumscribed about $\triangle ABC$.□

Theorem 4 *Assume, that the boundary of $\Omega \subset \mathbb{R}^2$ consists of a finite number of closed broken lines without multiple points, then the algorithm 2 leads to the Delaunay triangulation.*

Proof: when during the consecutive step of the advancing front technique, segment **AB** (Fig. 5) is chosen; then, the newly-created triangle **ABD** (the triangle **ABC** just exists in the current triangulation) has that property that the circle circumscribed about triangle **ABD** doesn't contain any point from the set of candidates (definiton 3) in its interior. As the triangulated domain is not convex, it may contain other points, but they are separated by the boundary of the domain. So the conditions of Delaunay triangulation of the domain are satisfied. According to algorithm 2, point **D** does not belong to the circle circumscribed about triangle **ABC**, the analogous situation happens when sides **AD** or **BD** of triangle **ABD** lie on the front (they are sides of the triangles of the current triangulation or boundary segments). Taking into account the lemma 1, the set of triangles obtained by adding **ABD** to it is still the Delaunay triangulation. The application of mathematical induction finishes the proof. □

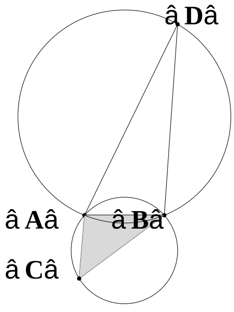

Figure 5. Ilustration of lemma 1.

4. The Delaunay-like triangulation of clouds of points on the surface

The theorems of section 3 allow us to define the triangulation of a cloud of points defining a surface. In this section, the generalization of algorithm 2 onto the set of points of cloud representing surface S is presented. Like in section 3, a set of internal points Λ and a set of boundary points Γ are distinguished.

Definition 4 *A set of triangles $\mathcal{T} = \{T_k\}_{k=1}^{N_T}$ is called a surface S semi-Delaunay triangulation [5], if:*

a) *each triangle from the set $\mathcal{T}$ has all his vertices from the set $\Gamma \bigcup \Lambda$,*

b) $\forall\, i, j \;\; 1 \leqslant i, j \leqslant N_T$ *and* $i \neq j$, *then* $T_i \neq T_j$,

c) $\forall\, T \in \mathcal{T}$ *and for every edge* $\mathbf{PQ} \in \overline{T}$ *the triangle T is the only triangle of the set $\mathcal{T}$ having this edge* $\mathbf{PQ}$ *or there exists exactly one triangle* $T' \in \mathcal{T}$, $T' \neq T$, *such that* $\mathbf{PQ} \subset \overline{T'}$,

d) *For every pair of triangles $T_1, T_2 \in \mathcal{T}$ $T_1 \neq T_2$ sharing a common edge, at least for one triangle of the pair, the sphere passing through its vertices and having the center lying in the plane of the triangle does not contain any vertices of the other triangle in its interior.*

e) $\forall\, \mathbf{P} \in \Lambda \bigcup \Gamma \;\exists T \in \mathcal{T}$, *such, that* $\mathbf{P} \in \overline{T}$.

Points a, b, e of the definition 4 are obvious. Point c is the well-known compatibility condition [14, 15]. Point d is introduced by the author [5], and is connected with the Delaunay triangulation by the following theorem:

Theorem 5 *It is assumed, that S is a surface contained on a plane surface contained in $3-D$ space, let Λ be a set of internal points and Γ a set of boundary points of surface S. If a set of triangles $\mathcal{T} = \{T_k\}_{k=1}^{N_T}$ is the semi-Delaunay triangulation of the surface S according to the definition 4, then set $\mathcal{T}$ is the Delaunay triangulation of surface S treated as the plane domain.*

Proof: the condition d) of the definition 4 means that all spheres passing through the vertices of triangles $T \in \mathcal{T}$ and the centers lying in the plane of $T \in \mathcal{T}$ lie on the plane surface containing S. According to definition 4, for any $T, T' \in \mathcal{T}$ $T \neq T'$, the sphere passing through the vertices of one of them does not contain any vertices of the other in its interior. It is obvious that the circumferences obtained as a result of the intersection of spheres passing through the vertices of triangles $T \in \mathcal{T}$, with the centers lying in the plane of $T \in \mathcal{T}$. The considered plane has the following property: $\forall\ T_i, T_j \in \mathcal{T}\ K_{T_i}$ sharing a common edge K_{T_i} do not contain vertices of T_j in its interior **or** vice-versa. The application of the lemma 1 finishes the proof. □

***Remark** 2 Theorem 5 motivates us to call this type as a generalization of Delaunay triangulation onto surfaces. The problem of its optimality is an open problem, but if the points are very close to each other, then locally a tangent plane surface is very close to the considered surface, and according to theorem 5, is very close to the optimal triangulation.*

5. Algorithm of semi-Delaunay surface triangulation

In this section, the algorithm of triangulation of a cloud of points on the orientable surface S is presented. It is a generalization of algorithm 2. Similarly to section 4, the set internal points Λ and boundary points Γ are introduced. It is assumed that points are so enumerated that, when during moving in accordance with the growing numbering, the surface leaves on the left. The surface may be multi-connected with a finite number of loops. The surface is oriented by the vector product of two vectors denoted by the triangle orientation adjacent to the current front segment (Fig. 6a). In the case of a boundary segment, the normal vector is denoted by the vector product of two consecutive boundary straight-line segments treated as vectors (Fig. 6b).

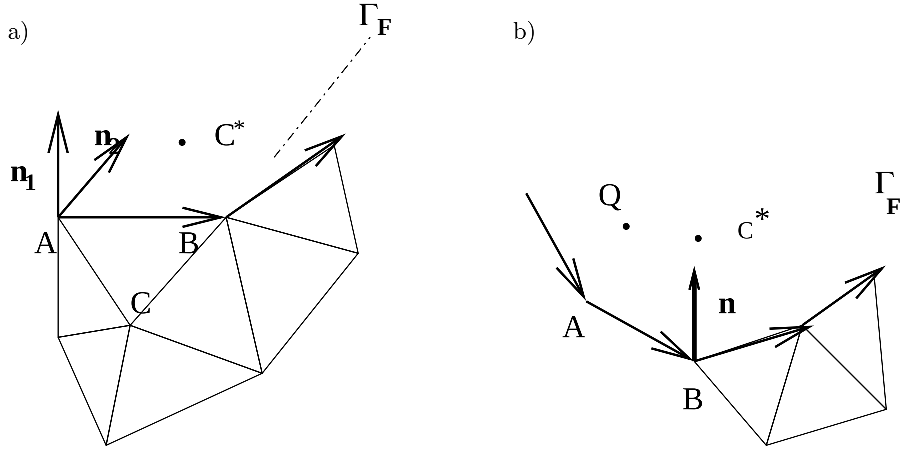

Figure 6. a) **AB** — side of the triangle; b) **AB** — the boundary segment.

By analogy to the 2-D case, front Γ_F is introduced. At the beginning of triangulation, the front consists of boundary segments. During triangulation, the front is being modified until becomes an empty set.

Consider an arbitrary front Γ_F segment $\mathbf{AB}$; the following definition is introduced:

Definition 5 *Point $\mathbf{C}^*$ lying on front Γ_F^* or belonging to set $\Lambda^* \subset \Lambda$, where set Λ^* is the set of points from set Λ (which do not form vertices in any current triangles) is called a candidate if it satisfies the following condition:*

$$(\mathbf{AC}^*, \mathbf{n}_2) > 0, \quad \textit{where} \quad \mathbf{n}_1 = \mathbf{AC} \times \mathbf{AB}, \quad \mathbf{n}_2 = \mathbf{n}_1 \times \mathbf{AB} \tag{14}$$

and triangle $\triangle\mathbf{ABC}$ is the triangle obtained during triangulation (Fig. 6a). In the case when $\mathbf{AB}$ is a boundary segment (Fig. 6b), the found point $\mathbf{Q} \in \Lambda$ is the closest to segment $\mathbf{AB}$. Vector $\mathbf{n}_1 = \mathbf{AB} \times \mathbf{AQ}$ and $\mathbf{n}_2 = \mathbf{n}_1 \times \mathbf{AB}$.

For the considered front segment $\mathbf{AB}$, the set of candidates $U = \{\mathbf{C}_i\}_{i=1}^{N_U} \subset \Gamma_F \bigcup \Gamma^*$ is defined as a set of points satisfying definition 5 and lying in the ball passing through points $\mathbf{A}$ and $\mathbf{B}$ with radius R (usually $R = ||\mathbf{AB}||$). The center of the ball is contained in the plane of the triangle $\mathbf{ABC}$ of the existing triangulation or in the plane of the triangle $\mathbf{ABQ}$ in the case when $\mathbf{AB}$ is a boundary segment, where $\mathbf{Q}$ is the end of the neighboring to $\mathbf{AB}$ boundary segment.

From set U, point $\mathbf{C}^*$ satisfying condition 15 is selected.

$$\forall\, \mathbf{D} \in U \qquad \mathbf{C}^* \in B_{\mathbf{ABD}} \tag{15}$$

where $B_{\mathbf{ABD}}$ (Fig. 7a) is the ball whose sphere passes through points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$ and the center lying in the plane of triangle $\mathbf{ABC}$.

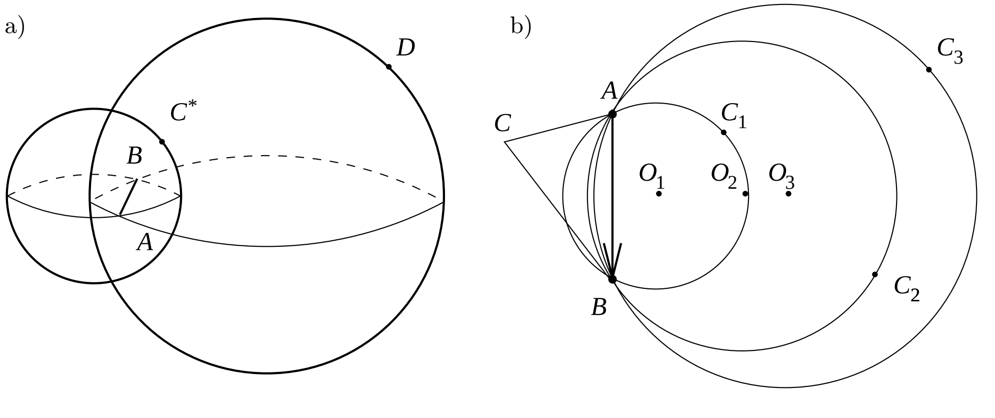

Figure 7. a) $\mathbf{AB}$ — the candidate choice; b) $\mathbf{AB}$ — The centers of balls $B_{\mathbf{ABD}}$ for $\mathbf{D} \in U$.

It can be observed that the centers of the balls used in definition 5 lie on a straight line perpendicular to segment $\mathbf{AB}$ and passing through the center of $\mathbf{AB}$ (Fig. 7b).

To satisfy condition 15, the following relation in set U is introduced: (Fig. 7):

$$\mathbf{C}_1 \preceq \mathbf{C}_2 \Longleftrightarrow \mathbf{C}_1 \in B_{AB\mathbf{C}_2} \tag{16}$$

Relation $\preceq$ defined by 16 has the following properties:

(i) $\forall\, \mathbf{C}_1,\, \mathbf{C}_2 \in U\ \mathbf{C}_1 \preceq \mathbf{C}_2$ or $\mathbf{C}_2 \preceq \mathbf{C}_1$ (totality),
(ii) If $\mathbf{C}_1 \preceq \mathbf{C}_2 \wedge \mathbf{C}_2 \preceq \mathbf{C}_1$, then $B_{\mathbf{ABC}_1} = B_{ABC_2}$,
(iii) $\forall\, \mathbf{C} \in U \qquad \mathbf{C} \preceq \mathbf{C}$ (reflexivity),
(iv) $\forall\, \mathbf{C}_1,\, \mathbf{C}_2,\, \mathbf{C}_3, \in U,\quad$ if $\quad \mathbf{C}_1 \preceq \mathbf{C}_2 \wedge \mathbf{C}_2 \preceq \mathbf{C}_3 \Longrightarrow \mathbf{C}_1 \preceq \mathbf{C}_3$ (transitivity).

It will be shown how to fast check the condition of relation $\preceq$. Let the system of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be the local coordinate system at the origin, at point $\mathbf{O} = \frac{1}{2}(\mathbf{A} + \mathbf{B})$, defined as follows:

$$\mathbf{u}_1 = \frac{\mathbf{AB}}{||\mathbf{AB}||}, \quad \mathbf{u}_2 = \frac{\mathbf{n}_2}{||\mathbf{n}_2||}, \quad \mathbf{u}_3 = \frac{\mathbf{n}_1}{||\mathbf{n}_1||}. \tag{17}$$

By these notations, the following theorem is true:

Theorem 6

$$\mathbf{C}_1 \preceq \mathbf{C}_2 \Longleftrightarrow (\mathbf{OO}_1, \mathbf{u}_2) \leqslant (\mathbf{OO}_2, \mathbf{u}_2), \tag{18}$$

where $\mathbf{O}_1$ *i* $\mathbf{O}_2$ *are appropriately the centers of the balls* $B_{\mathbf{ABC}_1}$ *and* $B_{\mathbf{ABC}_2}$*.*

Proof: on the assumption that the centers of balls $B_{\mathbf{ABC}_i}$ for $i = 1, 2$ lie in the plane denoted by the vertices of triangles $\triangle\mathbf{ABC}_i$, the spheres of these balls contain points $\mathbf{A}$, $\mathbf{B}$, the intersections of those spheres is the circle lying in the plane perpendicular to the plane passing through points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ with the center at the half of the straight line segment $\mathbf{AB}$ and the diameter equal to the length of $\mathbf{AB}$. The centers of balls $B_{\mathbf{ABC}_i}$ lie on a straight line perpendicular to the plane denoted by the circle being the intersection of these spheres and passing through the center of the circle. That means that centers O_1, O_2, $O_3, \ldots$ of balls $B_{\mathbf{ABC}_1}$, $B_{\mathbf{ABC}_2}$, $B_{\mathbf{ABC}_3} \cdots$ lie on a straight line passing through the center of the straight line segment $\mathbf{AB}$. The intersection of the plane of triangle $\mathbf{ABC}$ with ball $B_{\mathbf{ABC}_i}$ is a circle with center D_i (Fig. 7b). If $\mathbf{C}_2 \in B_{\mathbf{ABC}_1}$, then the center of $B_{\mathbf{ABC}_2}$ is moved to the left with respect to the center of $B_{\mathbf{ABC}_1}$, which completes the proof.

Theorem 6 means that the coordinate of $\mathbf{OO}_1$ associated with basis vector $\mathbf{u}_2$ is not greater than the the corresponding coordinate of $\mathbf{OO}_2$.

The introduced relation $\preceq$ is not yet an ordering relation, because the antisymmetry condition is still not satisfied. Practically, it means that the minimal element always exists, but unfortunately, it is not defined uniquely. For the sake of the unique definition of the minimal element, the following relation is introduced:

Definition 6 *Two arbitrary* $\mathbf{C}_1$, $\mathbf{C}_2 \in U$ *are in the relation* $\cong$*, if:*

$$\mathbf{C}_1 \cong \mathbf{C}_2 \Longleftrightarrow B_{\mathbf{ABC}_1} = B_{\mathbf{ABC}_2} \tag{19}$$

It can be checked that relation $\cong$ is an equivalence relation. An equivalence class of the relation is a set of points from U lying on the sphere passing through points $\mathbf{A}$ i $\mathbf{B}$, with the center lying in the plane of triangle $\triangle\mathbf{ABC}$.

Quotient set:

$$U^* = U/\cong \qquad \text{with the following relation:} \tag{20}$$

$$[\mathbf{C}_1] \preceq_* [\mathbf{C}_2] \Longleftrightarrow \mathbf{C}_1 \in B_{AB\mathbf{C}_2} \tag{21}$$

is an ordered set.

The satisfaction of the relation $\preceq_*$ doesn't depend on the choice of a representative of a given class, and all of the conditions of an ordering relation are satisfied. The lowest element according to this relation is an equivalence class which may contain more than one element. In this case we have a singular case (by the analogy to $2-D$ case); otherwise, the selection of a point to create a triangle on front segment $\mathbf{AB}$ is finished. In the singular case the points having the largest coordinate corresponding to basis vector $\mathbf{n}_2$ are selected. From these points that one is chosen which coordinate corresponding to basis vector $\mathbf{n}_3$ is the smallest.

After the point $\mathbf{C}^* \in U$ selection, which is to create a new triangle with vertices $\mathbf{A}$, $\mathbf{B}$, it is necessary to check whether front Γ_F is intersected by the two strips of planes perpendicular to the plane of the triangle $\mathbf{ABC}^*$ (Fig. 8b) the first passing through points $\mathbf{A}$, $\mathbf{C}^*$ and the second through points $\mathbf{B}$, $\mathbf{C}^*$. These two strips are bounded by the two straight lines passing appropriately through the points $\mathbf{A}$, $\mathbf{C}^*$ and $\mathbf{B}$, $\mathbf{C}^*$.

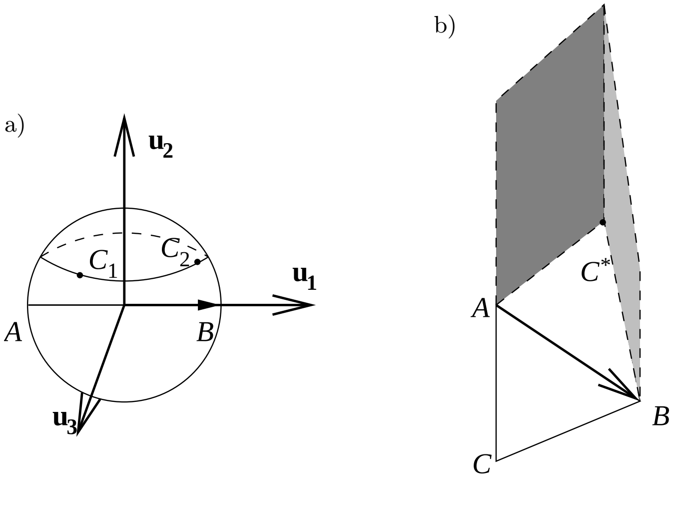

Figure 8. a) Points C_1 i C_2 with that same coordinate with respect to $\mathbf{u}_2$; b) Intersection of the bands of the planes with the front.

If there are no intersections, the following condition is checked:

$$\{\mathbf{C}^* A\} \bigcap \Gamma_F \in \{\{\emptyset\}, \{\mathbf{A}\}, \{\mathbf{C}^*, \mathbf{A}\}, \{\mathbf{C}^* A\}\} \tag{22}$$

$$\{\mathbf{BC}^*\} \bigcap \Gamma_F \in \{\{\emptyset\}, \{\mathbf{B}\}, \{\mathbf{C}^*, \mathbf{B}\}, \{\mathbf{BC}^*\}\} \tag{23}$$

The whole algorithm of the semi-Delaunay surface cloud points can be written in the following form:

Algorithm 3

1. $\Gamma_F \Longleftarrow$ *the set of boundary segments denoted by boundary points and defining positive orientation with respect to the surface.*
2. *IF*($\Gamma_F \neq \emptyset$) *THEN*
   find the lowest segment $\mathbf{AB}$ *belonging to* Γ_F.
   ELSE
   finish the triangulation
   ENDIF.
3. $R \Longleftarrow ||\mathbf{AB}||$.
4. *appoint the set of candidates* $U \subset \Lambda^* \bigcup \Gamma_F$ *belonging to ball with the center lying in the plane of triangle* $\mathbf{ABC}$ *with radius* r *and the sphere passing through points* $\mathbf{A}$, $\mathbf{B}$ *and satisfying condition 14.*
5. *find the subset of the points* $\mathbf{C}^* \in U_1 \subset U$ *satisfying condition:*

$$\mathbf{C}^* \in U \quad \forall \mathbf{D} \in U, \quad \mathbf{C}^* \preceq \mathbf{D}. \tag{24}$$

6. *If* $\#U_1 = 1$ *(*$\#X$ *denotes the number of elements of set* X*), go to 8.*
7. *Find the set* $U_2 \subset U_1$ *about the maximal coordinate corresponding to the basis vector* $\mathbf{u}_2$
   IF($\#U_2 = 1$) *go to 8*
   ELSE
   Determine set $U_3 \subset U_2$ *of points with the minimal coordinate corresponding to the vector* $\mathbf{u}_3$ *and accept any point from the set* U_3 *as the searched point* $\mathbf{C}^*$.
8. *If the appropriate plane bands lying in the perpendicular planes to the triangle* $\mathbf{ABC}$ *plane and passing through the points* $\mathbf{A}, \mathbf{C}^*$ *and* $\mathbf{B}, \mathbf{C}^*$ *as in figure 8b don't intersect front* Γ_F
   then
   a) *add the triangle* $\triangle\mathbf{ABC}$ *to the triangles list,*
   b) *modify the front* Γ.

otherwise
$U \Longleftarrow U \setminus \{\mathbf{C}^*\}$
IF($U \neq \emptyset$) THEN
go to 6
ELSE
$R \Longleftarrow R + R$
go to 5
ENDIF
ENDIF.

6. The numerical examples

Algorithm 3 was implemented, and its results are given now. The computer program GRID3-S was created and tested, and some numerical results are presented. Most of the examples are surfaces of revolution, so the points were equidistantly generated on parallel circles. The surfaces were split into a few pieces to distinguish internal and boundary points.

Figures 9, 10 present torus covered with a mesh about the constant size $\rho = 0.35$. Figure 10 illustrates how algorithm 3 functions.

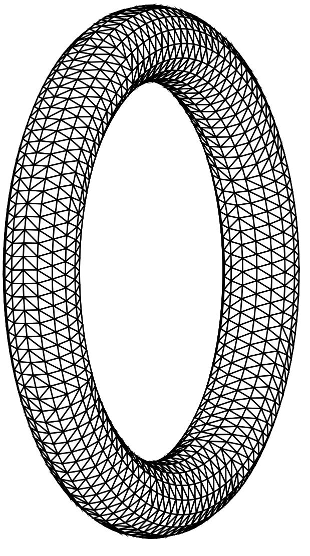

Figure 9. Torus covered with the triangular mesh

Figure 11 presents the cylinder connected with the part of the sphere, the assumed mesh size was $\rho = 0.09$. Figure 12 presents the cylinder connected with cubicoid, the assumed mesh size was $\rho = 0.15$.

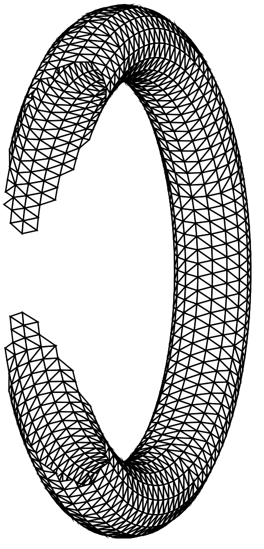

Figure 10. Part of the torus illustrating the method of triangulation.

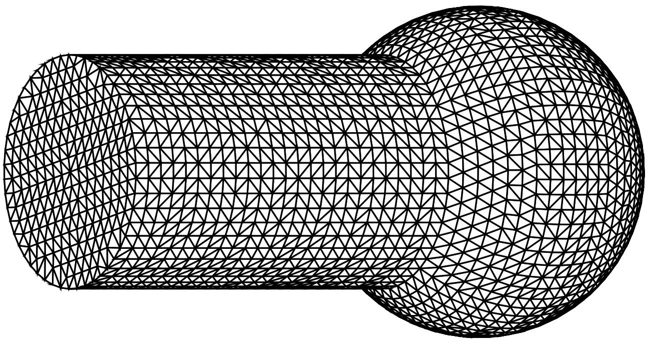

Figure 11. Cylinder connected with the part of sphere.

In Figures 11, 12, it can be observed how different types of surfaces are connected. In this case, the points are generated on the common curves of those surfaces.

The meshes in Figures 9 and 10 are created on the surfaces of the same types, the triangles formed at common curves are well conditioned, as is the whole mesh.

Figure 13 shows the triangulated cap of a sphere. The cap of the sphere was divided into 3 patches consisting of spherical triangles. Each spherical cap is oriented by the order of its vertices regarding the surface orientation with respect to the external normal (like in a 2-D domain, where the external normal to $x - y$ plane is the versor of z axis). It is important to keep that same surface orientation for each

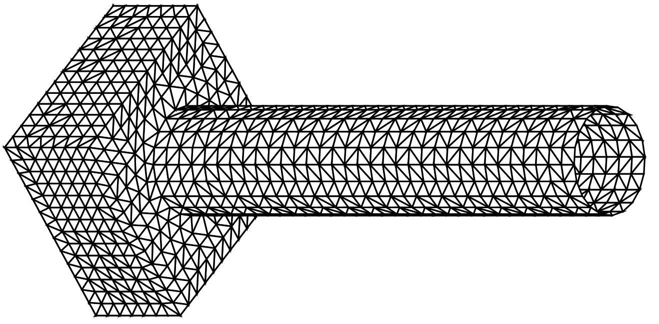

Figure 12. Cylinder connected with the cuboid.

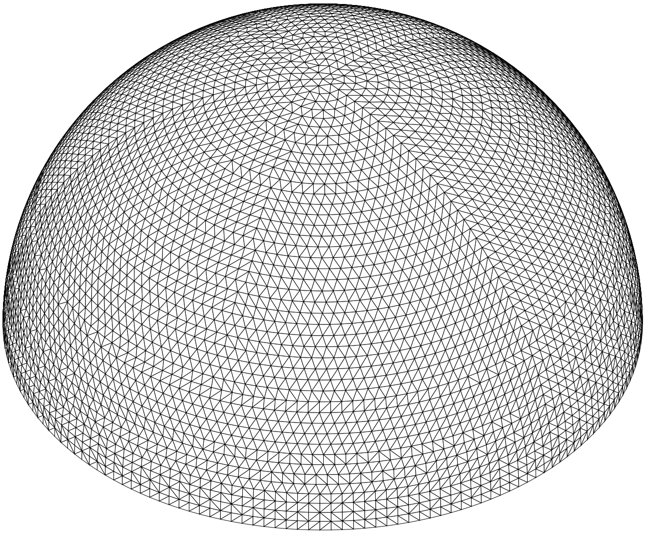

Figure 13. The cap of sphere covered with 10279 triangles.

patch. The enumeration of points on boundaries of each patch is described by the enumeration of its vertices.

The numerical analysis of computational complexity is shown in Table 1. The simulations were performed on a DUAL CORE INTEL 2×2,4 GHz with 3 GB RAM.

Table 1
The dependence between number of triangles and CPU time for the surface from Figure 13.

Number of triangles	326	530	1286	4963	7779	10279
CPU TIME	0.006s	0.007s	0.013s	0.038	0.051	0.0905

7. Summary and closure

In this paper, an algorithm of unstructured grid generation of clouds [12, 16] of points on a surface is provided. The paper contains a theoretical background, the formal definition of the semi-Delaunay property, and the triangulation algorithm of points scattered on the surface. It was proven that the proposed definition is a correct generalization of $2-D$ Delaunay algorithm of coupling Delaunay approach (coupled with the advancing front technique). The surface algorithm was implemented, and numerical results were obtained.

The original elements of the paper are:

- The definition of the semi-Delaunay triangulation of clouds of points on a surface of any shape.
- Theorem 2–6 and the lemma 1.
- Algorithm 3 of combining Advancing Front Method and semi-Delaunay Triangulation of clouds of points given on a surface of any shape.
- The computer code and numerical results.

The proposed approach can be applied to the triangulation of disjoint, multipatch surfaces, (even in close proximity to each other).

References

[1] Delaunay P. B.: Sur la Sphère Vide, *Bulletin de l'Académie des Sciences de l'URSS, Classe de Sciences Mathématiques et Naturelles*, pp. 793–800, 1934.

[2] Ebeida M. S., Mitchell S. A., Davidson A. A., Patney A., Knupp P. M., Owens J. D.: Efficient and good Delaunay meshes from random points. *Computer-Aided Design*, **43**, 6, 1506–1515, 2011.

[3] Guan Z., Shan J., Zheng Y., Gu Y.: An extended advancing front technique for closed surfaces mesh generation. *Int. J. Num. Meth. Engng*, **74**, pp. 642–667, 2008.

[4] Jin H., Wiberg N. E.: Two-dimensional mesh generation, adaptive remeshing and refinement. *Int. J. Num. Meth. Engng*, **29**, pp. 1501–1526, 1990.

[5] Kucwaj J.: Delaunay Triangulation of Surfaces. *ZAMM*, 76, p. 3, 249, 250, 1996.

[6] Kucwaj J.: Generation of Hybrid Grids over Plane Domains. *Computer Assisted Mechanics and Engineering Sciences*, **7**, pp. 607–614, 2000.

[7] Kucwaj J.: Glossary of grid generation, in *Handbook of grid generation.* Edited by J. F. Thompson, B. K. Soni, N. P. Weatherwill, Boca Raton London New York Washington, D. C., Appendix A-17 & Appendix B-29. *Computer Assisted Mechanics and Engineering Sciences*, **7**, pp. 607–614, 2000.

[8] Lo S. H.: Delaunay triangulation of non-convex planar domains. *Int. J. Num. Meth. Engng*, **28**, pp. 2695–2707, 1989.

[9] Lo S. H. : Finite element mesh generation and adaptive meshing. *Progress in Structural Engineering and Materials*, **4**, pp. 381–399, 2002.

[10] Lo S. H.: A new mesh generation scheme for arbitrary planar domains. *Int. J. Num. Meth. Engng*, **21**, pp. 1403–1426, 1985.

[11] Loehner R., Morgan K.: An unstructured multigrid method for elliptic problems. *Int. J. Num. Meth. Engng*, **24**, pp. 101–115, 1987.

[12] Ohrhallinger S., Mudur S. P.: Interpolating an unorganized 2D point cloud with a single closed shape. *Computer-Aided Design*, **43**, pp. 1629–1638, 2011.

[13] Rajan V.T.: Optimality of the Delaunay triangulation in $\mathbb{R}^d$. *Discrete Comput. Geom.*, **12**, pp. 189–202, 1994.

[14] Schroeder W. J., Shepard M. S., Geometry-based fully automatic mesh generation and the Delaunay triangulation. *Int. J. Num. Meth. Engng*, **26**, pp. 2503–2515, 1988.

[15] Wang Y., Feng H. Y., Delorme F. E., Engin S.: An adaptive normal estimation method scanned point cloud with small features. *Computer Aided Design*, **45**, pp. 1333–1348, 2013.

[16] Hsi-Yung Feng, Endrias D. H., Taher M. A.: Hao Song, An accurate and efficient algorithm for determining minimum circumscribed circles and spheres from discrete data points. *Computer Aided Design*, **45**, pp. 105–112, 2013.

Affiliations

Jan Kucwaj
Institute for Computer Science, Department of Physics, Mathematics and Computer Science, Cracow University of Technology, ul. Warszawska 24, 31-155 Kraków, jkucwaj@pk.edu.pl

Received: 17.02.2014
Revised: 13.04.2014
Accepted: 9.06.2014