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## INFERENCES ON UNIFIED PROGRESSIVE HYBRID CENSORING UTILIZING WEIBULL-HALF-LOGISTIC DISTRIBUTION WITH APPLICATION TO MICRO DROP DATA

**Abstract** *In this paper, we have delved into the application of classical and Bayesian approaches for the Weibull-half-logistic distribution based on unified progressive hybrid censoring scheme. Maximum likelihood estimation is employed for the two parameters, and the establishment of approximate confidence bounds is based on asymptotic normality. The Lindley's approximation and Markov chain Monte Carlo technique from the squared error and Linex loss functions are performed to obtain the symmetric and asymmetric Bayesian inferences. The credible bounds of the unknown parameters are also provided. The performance of the suggested technique and estimators has been assessed and compared using the outcomes of simulated data with the censoring strategies utilized. Finally, for illustrative purposes, the virus-containing micro drop data under simultaneous airflow with velocity of  $0.2^{m/s}$  is examined.*

**Keywords** Bayesian Inference, Unified Progressive Hybrid, Virus-Containing Drops, Weibull-half-logistic Distribution.

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## 1. Introduction

The failure times of experimental items are commonly unavailable in reliability investigations or engineering tests. Censoring has become a commonly used phenomenon in survival analysis and reliability studies. A variety of censoring schemes have already been put forth by several researchers in order to improve the efficiency of an experiment. Type-I and Type-II censoring are the most popular censoring techniques. Childs et al. [7] delve into hybrid censoring scheme. Many studies considered the Type-I, Type-II, hybrid and their generalization censoring schemes. For instance, see ([3]- [9]- [13]- [16]- [15]- [19]- [21]). In the field of reliability tests, engineers prefer the progressive censoring scheme (*ProgresCeSc*) over the aforementioned censoring schemes (*CeSc*) due to the flexibility of this scheme in removing test items. The *ProgresCeS* is the most often used of these censoring schemes since it minimizes the cost and length of an experiment (see, [1]- [5]- [8]- [20]). To improve the efficiency of the experiments, progressive hybrid censoring scheme which is a combination of the conventional hybrid and progressive censoring schemes has been introduced. The drawback of this censoring is that the effective sample size may become zero, which is as similar as Type-I censoring. So, Gorny and Cramer [10] introduced the unified progressive hybrid censoring scheme (*UnPrHyCeSc*). Presume that  $n, m, k; (k < m), R = (R_1, R_2, \dots, R_m); \sum_{i=1}^m R_i = n - m$  and  $0 < T_1 < T_2$ , are the preassigned thresholds  $T_i; i = 1, 2$ . This experiment will be terminated at  $T^* = \max \{ \min \{ X_{k:m:n}, T_2 \}, \min \{ X_{m:m:n}, T_1 \} \}$ . Below are the different *UnPrHyCeSc* Cases:

Case I:  $\{X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}\}$  if  $X_{1:m:n} < T_1$ ,

Case II:  $\{X_{1:m:n}, \dots, X_{k:m:n}, \dots, X_{q_1:m:n}\}$  if  $X_{k:m:n} < T_1 < X_{m:m:n}$ ,

Case III:  $\{X_{1:m:n}, \dots, X_{k:m:n}\}$  if  $T_1 < X_{k:m:n} < T_2$ ,

Case III:  $\{X_{1:m:n}, \dots, X_{q_2:m:n}\}$  if  $T_2 < X_{k:m:n}$ .

where  $X$  is random variable and  $q_i$  denotes the number of failure units before  $T_i$ , for  $i = 1, 2$ . From the suggested *UnPrHyCeSc*, many *CeSc* can be introduced as follows: Progressive Type I hybrid censoring when  $T_1 \rightarrow 0$ .

Progressive Type II hybrid censoring when  $T_2 \rightarrow \infty$ .

Type I hybrid censoring when  $T_1 \rightarrow 0, R_1 = R_2 = \dots = R_{m-1} = 0, R_m = n - m$ .

Type II hybrid censoring when  $T_2 \rightarrow \infty, R_1 = R_2 = \dots = R_{m-1} = 0, R_m = n - m$ .

Type I censoring when  $T_1 = 0, m = 1, R_1 = R_2 = \dots = R_{m-1} = 0, R_m = n - m$ .

Type II censoring when  $T_1 = 0, T_2 \rightarrow \infty, m = 1, R_1 = R_2 = \dots = R_{m-1} = 0, R_m = n - m$ .

Unified hybrid censoring when  $R_1 = R_2 = \dots = R_{m-1} = 0, R_m = n - m$ .

Moreover, in practical studies, a large number of statistical distributions have been proposed for analyzing real-world data across various applied fields. However, it is not possible to model many complex data by well-known standard distributions. The half-logistic (HL) model, introduced by Balakrishnan [4] using the absolute transformation of the logistic distribution, is highly regarded in modeling

datasets across a wide range of fields and holds significant importance in statistics, physics, hydrology, and logistic regression. Let  $F_X(x)$  denotes a continuous distribution, and consider the Weibull cumulative distribution function (CDF) given by  $F_X(x) = 1 - \exp(-\zeta x^\nu)$ ;  $x > 0, \nu, \zeta > 0$ . By replacing  $x$  with  $J(x)/(1 - J(x))$ , a new family of cumulative distribution function can be formulated as follows:

$$F_X(x | \zeta, \nu, \lambda) = 1 - \exp\left(-\zeta\left(\frac{J(x; \lambda)}{1 - J(x; \lambda)}\right)^\nu\right), x > 0, \zeta, \nu > 0,$$

where  $J(x; \lambda)$  represents a baseline cumulative distribution function that depends on a parameter vector  $\lambda$ . From this point forward, assume that  $J$  is a continuous baseline distribution. For each  $J$  distribution, the Weibull- $J$  distribution, characterized by the inclusion of the two parameters  $\zeta$  and  $\nu$ , is defined by the CDF above (see, Bourguignon et al. [6]). Specifically, for the half-logistic distribution,  $J(x)/(1 - J(x))$  simplifies to  $(e^x - 1)/2$ . For the half-logistic distribution, the readers can follow the articles by Kazemi and Kohansal [12] and Bourguignon et al. [6].

Considering the balance between model complexity and flexibility, we propose the two-parameter Weibull-half-logistic (*WeHL*) distribution. The probability density function (pdf), cumulative distribution function (cdf) and the hazard function (haf) of *WeHL* distribution are shown in (1), (2) and (3), respectively.

$$f(x; \zeta, \nu) = \zeta \nu 2^{-\nu} (e^x - 1)^{\zeta-1} e^{x-\nu 2^{-\zeta} (e^x - 1)^\zeta}; \quad x > 0, \nu, \zeta > 0, \quad (1)$$

$$F(x; \zeta, \nu) = 1 - e^{-\nu 2^{-\zeta} (e^x - 1)^\zeta}; \quad x > 0, \quad (2)$$

and

$$H(x; \zeta, \nu) = \nu \zeta 2^{-\zeta} e^x (e^x - 1)^{\zeta-1}; \quad x > 0. \quad (3)$$

Here,  $\nu$  and  $\zeta$  are known as the scale and shape parameters respectively. Various shapes of the probability density, reliability and hazard functions of the *WeHL* are depicted in Figure 1. The *WeHL* distribution is the most commonly used distribution in both academia and practice. This distribution exhibits decreasing or increasing constant failure rates and is shown to fit many real life situations better than the conventional Weibull or Lomax distributions. Though several studies have given the suggested censoring a mathematical treatment, no attempt has been made to evaluate *WeHL* model parameters based on the *UnPrHyCeSc*. Consequently, in the present work, the statistical inference issue of *WeHL* distribution and its engineering application is addressed. In the main results, maximum likelihood estimates (MLEs) are observed to be inaccessible in closed form. In order to obtain MLEs, a suitable numerical method has been applied. Gamma priors with squared and Linex error loss functions have been utilized to calculate the Bayes and highest posterior density interval estimations. The Lindley approach is proposed to Bayes point estimates. Also, to simulate Markovian samples from the same unknown parameters, the Metropolis-Hastings (MH) sampler has also been recommended. MCMC convergence is also discussed. The dataset of the virus-containing micro drop based on velocity of

$0.2^{m/s}$  is analyzed using the proposed methods. Section 2 developed the point and interval estimations via maximum likelihood approach. The existence and uniqueness of the MLEs have been studied. The Bayesian estimates using two approximations based on two loss functions are presented in Section 3. All theoretical outcomes are illustrated with simulation studies under various censoring plans in Section 4. The virus-containing micro droplets dataset is analyzed for illustration in Section 5. Finally, Section 6 concludes this study.

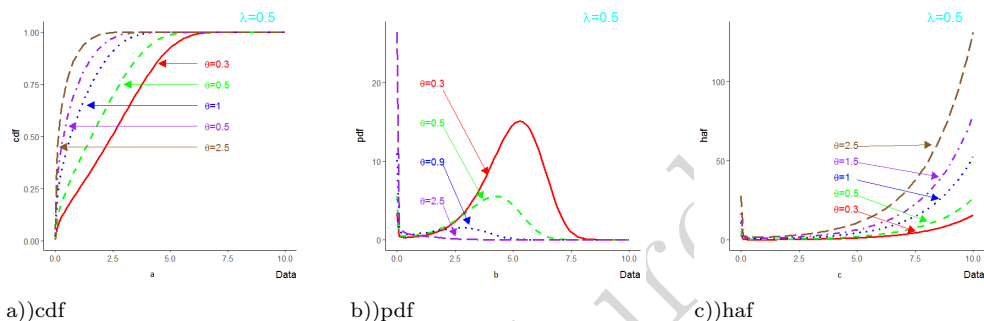


**Figure 1.** The graphs of the WeHL distribution.

## 2. Likelihood Setup

Let  $X_{1:m:n}, \dots, X_{D:m:n}$  be a unified progressive hybrid censored sample produced from the *WeHL* distribution in Equations (1) and (2), the likelihood function is:

$$\begin{aligned}
 L(\zeta, \nu \mid \underline{X}) &= \prod_{i=1}^D f(x_{i:m:n}; \zeta, \nu) [1 - F(x_{i:m:n}; \zeta, \nu)]^{R_i} [1 - F(T; \zeta, \nu)]^{R^*} \\
 &= \prod_{i=1}^D \zeta \nu 2^{-\zeta} w_{i:m:n}^{\zeta-1} e^{x_{i:m:n}} \left[ e^{-\nu 2^{-\zeta} w_{i:m:n}^{\zeta}} \right]^{R_i+1} \left[ e^{-\nu 2^{-\zeta} w_T^{\zeta}} \right]^{R^*}. \quad (4)
 \end{aligned}$$

Where,  $(D, T)$  are  $(m, x_m)$ ,  $(d_1, T_1)$ ,  $(k, x_k)$  and  $(d_2, T_2)$  for Case I, Case II, Case III and Case IV respectively. So, the log-likelihood becomes:

$$\begin{aligned}
 \ell(\zeta, \nu \mid \underline{X}) &= \ln L(\zeta, \nu \mid \underline{X}) = D \ln(\nu) + D \ln(\zeta) - D\zeta \ln 2 + \sum_{i=1}^D x_{i:m:n} \\
 &\quad + (\zeta - 1) \sum_{i=1}^D \ln w_{i:m:n} - \nu 2^{-\zeta} \left[ \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta} \right]. \quad (5)
 \end{aligned}$$

Here,  $w_{i:m:n} = (e^{x_{i:m:n}} - 1)$  and  $w_T = (e^T - 1)$ . The MLEs are the solution of the following system of non-linear equations:

$$\frac{\partial \ell(\zeta, \nu | \underline{X})}{\partial \zeta} = 0 \quad \text{and} \quad \frac{\partial \ell(\zeta, \nu | \underline{X})}{\partial \nu} = 0$$

As a result, the proposed MLEs can be acquired from the following:

$$\frac{\partial \ell}{\partial \nu} = \frac{D}{\nu} - 2^{-\zeta} \left[ \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta} \right] = 0, \quad (6)$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \zeta} = \frac{D}{\zeta} - D \ln 2 - \nu 2^{-\zeta} \left[ \ln(1/2) \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} \ln w_{i:m:n} \right. \\ \left. + R^* (\ln(1/2) w_T^{\zeta} + w_T^{\zeta} \ln w_T) \right] + \sum_{i=1}^D \ln w_{i:m:n} = 0. \end{aligned} \quad (7)$$

The closed form expressions of the MLEs are not obtainable from the non-linear equations given above. Therefore, it is suggested to use some numerical methods to obtain the MLEs numerically. The several numerical procedures are available with statistical software. For instance, the *optim()* function in R software provides various optimization methods including the Nelder-Mead and Newton-Raphson methods. The Newton-Raphson method is an iterative optimization algorithm used to find successively better approximations to the roots of a real-valued function. This method updates parameter estimates by considering how changes in the parameters affect the likelihood function, allowing for rapid convergence to the optimal values when the initial estimates are close to the true parameters. Also, the Nelder-Mead method is a popular optimization algorithm which is a direct search method and does not require derivatives of the objective function. This algorithm is iterative and consists of four main steps: reflection, expansion, contraction, and shrinkage. In each iteration, the algorithm evaluates the values of the function at the vertices of a simplex (a geometric figure with  $n + 1$  vertices in  $n$  dimensions) and moves the simplex through the parameter space to find the minimum or maximum of the function. Because it does not rely on derivatives, the Nelder-Mead method is particularly useful when the objective function is not differentiable or when derivative calculations are complex. By utilizing these numerical methods, one can efficiently estimate the MLEs of the parameters even when closed-form solutions are not available.

## 2.1. Existence and uniqueness of MLEs

In this subsection, we aim to establish the existence and uniqueness of the MLEs  $\hat{\zeta}$  and  $\hat{\nu}$  for the parameters of the WeHL distribution under unified progressive hybrid

censoring. Let  $\phi(\zeta) = \frac{\partial \ell}{\partial \zeta}$  and  $\nu, \zeta \in (0, \infty)$ , when  $\zeta$  is known, fixed, and positive,  $\hat{\nu}$  exists and is unique and finite as given by

$$\hat{\nu}(\zeta) = \frac{D}{2^{-\zeta} \left[ \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta} \right]}$$

when  $\nu$  is known, the second derivative of the log-likelihood function with respect to  $\zeta$  is negative  $l_{\zeta\zeta} < 0$ . This means that  $\ell(\zeta, \nu | \underline{X})$  in Equation (5) is strictly concave in  $\zeta$ , it follows that for fixed  $\nu > 0$ , if  $\hat{\zeta}$  exists, it is unique. To check the behavior of  $\phi(\zeta) = \frac{\partial \ell}{\partial \zeta}$ , we see as  $\zeta \rightarrow 0$ ,  $\phi(\zeta) \rightarrow \infty$ . This means, the derivative tends to positive infinity, indicating that the slope of the log-likelihood function becomes very steep and positive near  $\zeta = 0$ . Also, as  $\zeta \rightarrow \infty$ ,  $\phi(\zeta) \rightarrow -D \ln 2 + \sum_{i=1}^D \ln w_{i:m:n}$ . This

expression is finite. The term  $-D \ln 2$  is negative, and  $\sum_{i=1}^D \ln w_{i:m:n}$  depends on the data. For large  $\zeta$ ,  $\phi(\zeta)$  can become negative if  $0 < x_{i:m:n} < \ln 2$ . This condition ensures that  $\sum_{i=1}^D \ln w_{i:m:n}$  is negative, making  $\phi(\zeta)$  negative for large  $\zeta$ . Since  $\phi(\zeta)$  is continuous on  $(0, \infty)$  and changes sign from positive to negative, there exists at least one  $\hat{\zeta} > 0$  such that:

$$\phi(\hat{\zeta}) = 0$$

The strict concavity of  $\ell(\zeta, \nu | \underline{X})$  ensures that this root is unique and finite if and only if  $0 < x_{i:m:n} < \ln 2$ . Therefore, when  $\nu$  is known and positive,  $\hat{\zeta}$  exists, is unique, and finite under the specified condition. In the case where the two parameters  $\nu$  and  $\zeta$  are both unknown,  $\phi(\zeta)$  is continuous and decreases monotonically from positive infinity to a finite negative value as  $\zeta$  increases from 0 to  $\infty$ . So, There exists a unique  $\hat{\zeta} > 0$  such that  $\phi(\hat{\zeta}) = 0$ . Once  $\hat{\zeta}$  is found,  $\hat{\nu}$  can be computed using the relationship between the parameters (Equation (6)). By examining the behavior of the equation  $\hat{\zeta}$  on the positive real line  $(0, \infty)$ , where  $\phi(\zeta)$  is given in Equation (7), we have

$$\phi(\zeta) \rightarrow \infty, \text{ as } \zeta \rightarrow 0$$

$$\phi(\zeta) \rightarrow -D \ln 2 + \sum_{i=1}^D \ln w_{i:m:n}, \text{ as } \zeta \rightarrow \infty.$$

It follows that  $\phi(\zeta) < 0$  for  $0 < x_{i:m:n} < \ln 2$ . Therefore, on  $(0, \infty)$ ,  $\phi(\zeta)$  is a continuous function which decreases monotonically from positive to negative values. Hence there exists at least one finite positive root of the equation  $\phi(\zeta) = 0$ . Also,  $l_{\zeta\zeta} < 0$  for all  $\zeta > 0$ . Hence  $\hat{\zeta}$  will be unique. Thus  $\hat{\zeta}$  will exist and will be unique if  $0 < x_{i:m:n} < \ln 2$ . This concludes that MLE  $\hat{\zeta}$  and hence  $\hat{\nu}$  exists, is unique and finite.

## 2.2. Approximate Confidence Intervals

The MLEs are not determinable in closed forms, thus determining the exact distribution of the MLEs is challenging and even not possible in this case. Therefore, we estimate the asymptotic confidence intervals of the parameters using the large sample property of the MLE. Under some regularity conditions, we have,

$\hat{\theta}$  is asymptotically consistent, i.e.,  $\lim_{n \rightarrow \infty} P(|\hat{\theta} - \theta| > \varepsilon) = 0$ ;  $\theta = (\nu, \zeta)$ .

$\hat{\theta}$  is asymptotically unbiased, i.e.,

$$\lim_{n \rightarrow \infty} E(\hat{\theta}) = \theta.$$

The MLE ( $\hat{\theta}$ ) follow the normal distribution with mean  $\theta$  and  $\text{var}(\hat{\theta})$ .

In our case, the ML estimate of model parameters has bivariate normal distribution with mean  $(\nu, \zeta)$  and variance covariance matrix obtained from the inverse of  $I(\nu, \zeta)$ , where  $I^{-1}(\nu, \zeta)$  is the inverse of the Fisher's information matrix. To obtain  $(1-s)\%100$ ;  $0 < s < 1$  confidence intervals for the unknown parameters based on *UnPrHyCeSc*, we use the observed information matrix as:

$$\begin{pmatrix} -l_{\nu\nu} & -l_{\nu\zeta} \\ -l_{\zeta\nu} & -l_{\zeta\zeta} \end{pmatrix} = \begin{pmatrix} -\partial^2 l / \partial \nu^2 & -\partial^2 l / \partial \nu \partial \zeta \\ -\partial^2 l / \partial \zeta \partial \nu & -\partial^2 l / \partial \zeta^2 \end{pmatrix}. \quad (8)$$

The elements of the observed information matrix are obtained as,

$$\begin{aligned} l_{\nu\nu} &= -D/\nu^2, \\ l_{\zeta\zeta} &= \frac{-D}{\zeta^2} - \nu 2^{-\zeta} \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} \ln(2^{-1} w_{i:m:n}) + R^* w_T^{\zeta} \ln(w_T), \\ l_{\nu\zeta} &= -2^{-\zeta} \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} \ln(2^{-1} w_{i:m:n}) + R^* w_T^{\zeta} \ln(w_T) \end{aligned} \quad (9)$$

So, for given confidence level  $s$  then  $(1-s)\%100$  approximate confidence intervals of the model parameters are given:

$$\left[ \hat{\nu} - Z_{s/2} \sqrt{\text{var}(\hat{\nu})}, \hat{\nu} + Z_{s/2} \sqrt{\text{var}(\hat{\nu})} \right] \text{ and } \left[ \hat{\zeta} - Z_{s/2} \sqrt{\text{var}(\hat{\zeta})}, \hat{\zeta} + Z_{s/2} \sqrt{\text{var}(\hat{\zeta})} \right] \quad (10)$$

Here,  $Z_{s/2}$  is the upper  $(s/2)^{th}$  percentile of the standard normal distribution. Where,  $\text{var}(\hat{\nu})$  and  $\text{var}(\hat{\zeta})$  are the entries on the main diagonal of the asymptotic variance covariance matrix  $I^{-1}(\nu, \zeta)$ . The approximate confidence interval is based on asymptotic approximations, which means it relies on the assumption that as the sample size grows larger, the distribution of the sample statistic approaches the normal distribution. This allows for the use of standard confidence interval formulas even when the underlying distribution is unknown. These intervals are derived based on sample data and provide a practical approach to interval estimation when exact calculations are either too complex or computationally intensive.

### 3. Bayesian Setup

This part aims to develop the Bayes estimators of all *WeHL* parameters, as well as their related Bayesian bounds. The independent prior distributions for  $\zeta$  and  $\nu$  are assumed to follow  $\text{Gamma}(a, b)$  and  $\text{Gamma}(c, d)$  respectively. Gamma distribution is a flexible distribution and is frequently used as prior distribution in previous studies. So, the joint prior distribution of  $\zeta$  and  $\nu$  can be expressed as:

$$\pi(\zeta, \nu) \propto \zeta^{a-1} e^{-b\zeta} \nu^{c-1} e^{-d\nu}, \quad \zeta, \nu > 0, \quad a, b, c, d > 0,$$

By utilizing Bayes Theorem, the joint posterior distribution is obtained as  $\pi(\zeta, \nu \mid \underline{X}) = K \ell(\zeta, \nu \mid \underline{X}) \pi(\zeta, \nu)$ , where  $K$  is the normalizing constant. Here, the joint posterior distribution for  $\zeta$  and  $\nu$  can be formulated as:

$$\begin{aligned} \pi(\zeta, \nu \mid \underline{X}) &= \frac{L(\zeta, \nu \mid \underline{X}) \pi(\zeta, \nu)}{\int_0^\infty \int_0^\infty L(\zeta, \nu \mid \underline{X}) \pi(\zeta, \nu) d\zeta d\nu} \\ &= \frac{\zeta^{D+a-1} \nu^{D+c-1} 2^{-D\zeta} e^{-b\zeta-d\nu} \prod_{i=1}^D w_{i:m:n}^{\zeta-1} e^{x_{i:m:n}} [A_{i:m:n}]^{R_i+1} [A_T]^{R^*}}{\int_0^\infty \int_0^\infty \zeta^{D+a-1} \nu^{D+c-1} 2^{-D\zeta} e^{-b\zeta-d\nu} \prod_{i=1}^D w_{i:m:n}^{\zeta-1} e^{x_{i:m:n}} [A_{i:m:n}]^{R_i+1} [A_T]^{R^*} d\zeta d\nu}. \end{aligned}$$

here,  $A_{i:m:n} = e^{-\nu 2^{-\zeta} w_{i:m:n}^\zeta}$ ,  $A_T = e^{-\nu 2^{-\zeta} w_T^\zeta}$  and  $L(\zeta, \nu \mid \underline{X})$  represents the likelihood function, and  $\pi(\zeta, \nu)$  denotes the joint prior function. So, the joint posterior distribution of the parameters  $\nu$  and  $\zeta$  is acquired through the Bayes principle, as expressed below:

$$\begin{aligned} \pi(\zeta, \nu \mid \underline{X}) &\propto \nu^{a-1} e^{-b\nu} \zeta^{c-1} e^{-d\zeta} (\nu \zeta 2^{-\zeta})^D \times \left( \prod_{i=1}^D e^{x_{i:m:n} w_{i:m:n}^{\zeta-1}} \right) \\ &\quad \times e^{-\nu 2^{-\zeta} [\sum_{i=1}^D (R_i+1) w_{i:m:n}^\zeta + R^* w_T^\zeta]} \end{aligned} \quad (11)$$

Although sampling directly from the joint posterior distribution can be challenging, sampling from the full conditional distributions is often more feasible. The MCMC method such as Gibbs sampling applies this by reducing the complexity of joint sampling to a series of one-dimensional sampling tasks. So, conditional posteriors of  $\nu$  and  $\zeta$  can be written as:

$$\pi_1(\nu \mid \zeta, \underline{X}) = \nu^{D+a-1} \exp \left\{ -\nu \left( b + 2^{-\zeta} \left[ \sum_{i=1}^D (R_i+1) w_{i:m:n}^\zeta + R^* w_T^\zeta \right] \right) \right\}, \quad (12)$$

and

$$\pi_2(\zeta \mid \nu, \underline{X}) = \zeta^{D+c-1} 2^{-D\zeta} \left( \prod_{i=1}^D w_{i:m:n}^{\zeta-1} \right)$$

$$\times \exp \left\{ -\nu 2^{-\zeta} \left[ \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta} \right] - d\zeta \right\}. \quad (13)$$

In order to obtain Bayes estimates, the posterior expected value of a loss function has to be minimized. In such case, the choice of different loss function reflects different estimation error effects. Thus, we consider two loss functions, squared error and Linex loss functions:

The squared error (SE) loss function which is symmetric is given by:

$$L(\theta^*, \theta) = (\theta^* - \theta)^2.$$

The Linex loss function which is asymmetric is given by:

$$L(\theta^*, \theta) \propto e^{h(\theta^* - \theta)} - h(\theta^* - \theta) - 1; \quad h \neq 0.$$

The Bayesian estimates under SE and Linex functions are posterior mean of  $\theta$  and  $-\frac{1}{h} \log(E[e^{-h\theta} | \underline{X}])$  where  $E[\cdot | \underline{X}]$  is the expectation with respect to  $\pi(\zeta, \nu | \underline{X})$ . In the Linex loss function, when  $h$  is very small (close to zero), the Linex function acts similar to the SE function. For more clarity, different plots of Linex loss function under different parameter values ( $h$ ) are depicted in Figure 2.

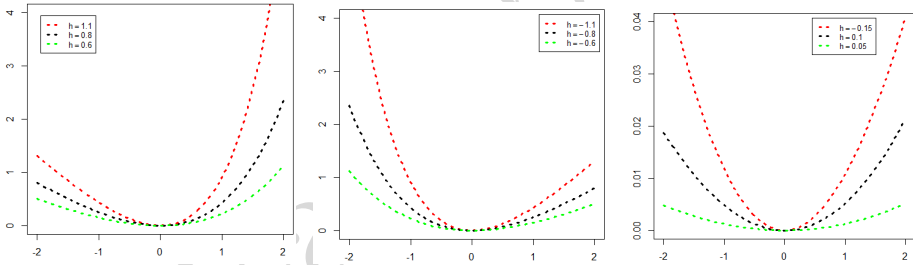


**Figure 2.** Linex loss function representations for different values of  $h$ .

The plots demonstrate that positive, negative and small values of  $h$  give more weight for overestimating, underestimating and almost equal estimation, respectively. It can be seen that as  $h$  towards zero, the function is approximately symmetric as the squared error loss function.

The Bayesian estimates under Linex and SE loss function can be written as  $\theta = (\nu, \zeta)$ :

$$\begin{aligned} \theta_{SE} &= E(\theta | \underline{X}) = \int_0^\infty \int_0^\infty \theta \nu^{a-1} e^{-b\nu} \zeta^{c-1} e^{-d\zeta} (\nu \zeta 2^{-\zeta})^D \\ &\times \left( \prod_{i=1}^D e^{x_{i:m:n} w_{i:m:n}^{\zeta-1}} \right) \times e^{-\nu 2^{-\zeta} [\sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta}]} d\zeta d\nu, \end{aligned}$$

and

$$\theta_{Linex} = -\frac{1}{h} \log(E[e^{-h\theta} | \underline{X}]) = -\frac{1}{h} \ln \left[ \int_0^\infty \int_0^\infty e^{-h\theta} \nu^{a-1} e^{-b\nu} \zeta^{c-1} e^{-d\zeta} (\nu \zeta 2^{-\zeta})^D \right]$$

$$\times \left( \prod_{i=1}^D e^{x_{i:m:n} w_{i:m:n}^{\zeta-1}} \right) \times e^{-\nu 2^{-\zeta} [\sum_{i=1}^D (R_i+1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta}] d\nu d\zeta}; \theta = (\nu, \zeta).$$

It's worth noting that, in many situations, explicit computation of Bayesian estimates is not feasible. The MCMC and Lindley's techniques can be applied to gain the approximate Bayes estimates. Further from the representation of the posterior density [11], it is noted that the conditional posterior of  $\nu$  follows Gamma distribution, whereas the posterior marginal distribution of parameter  $\zeta$  cannot refer to some well-known models.

### 3.1. Lindley's Approximation

In this subsection, we discuss the Bayes estimates of  $\nu$  and  $\zeta$  using Lindley's approximation ([14]) technique. We employ this method because the posterior distribution does not have a closed-form solution. When the posterior distribution is analytically intractable, exact estimates become computationally challenging or even impossible to calculate directly. This is typically due to the complexity of the likelihood function or the prior, making it difficult to derive an explicit posterior. So, the goal is to evaluate integrals that include the ratio of two integrals (Bayesian estimates) in the following form:

$$E[u(\nu, \zeta) | x_1, \dots, x_n] = \frac{\int_0^\infty \int_0^\infty u(\nu, \zeta) e^{L(\nu, \zeta) + p(\nu, \zeta)} d\nu d\zeta}{\int_0^\infty \int_0^\infty e^{L(\nu, \zeta) + p(\nu, \zeta)} d\nu d\zeta}.$$

According Lindley [14], and by defining  $p(\nu, \zeta) = \log \pi(\nu, \zeta)$ , the Lindley approximation used as:

$$\begin{aligned} E[u(\nu, \zeta) | x_1, \dots, x_n] &= \frac{\int_0^\infty \int_0^\infty u(\nu, \zeta) e^{L(\nu, \zeta) + p(\nu, \zeta)} d\nu d\zeta}{\int_0^\infty \int_0^\infty e^{L(\nu, \zeta) + p(\nu, \zeta)} d\nu d\zeta} \\ &= u(\nu, \zeta) + \frac{1}{2} [(u_{\zeta\zeta} + 2u_{\zeta p_{\zeta}}) \sigma_{\zeta\zeta} + (u_{\nu\zeta} + 2u_{\nu p_{\zeta}}) \sigma_{\nu\zeta} + (u_{\zeta\nu} + 2u_{\zeta p_{\nu}}) \sigma_{\zeta\nu} \\ &\quad + (u_{\nu\nu} + 2u_{\nu p_{\nu}}) \sigma_{\nu\nu}] + \frac{1}{2} [(u_{\zeta} \sigma_{\zeta\zeta} + u_{\nu} \sigma_{\zeta\nu}) (l_{\zeta\zeta\zeta} \sigma_{\zeta\zeta} + l_{\zeta\nu\zeta} \sigma_{\zeta\nu} + l_{\nu\zeta\zeta} \sigma_{\nu\zeta} \\ &\quad + l_{\nu\nu\zeta} \sigma_{\nu\nu}) + (u_{\zeta} \sigma_{\nu\zeta} + u_{\nu} \sigma_{\nu\nu}) (l_{\nu\zeta\zeta} \sigma_{\zeta\zeta} + l_{\zeta\nu\nu} \sigma_{\zeta\nu} + l_{\nu\zeta\nu} \sigma_{\nu\zeta} + l_{\nu\nu\nu} \sigma_{\nu\nu})], \end{aligned}$$

where  $u(\cdot)$  is arbitrary functions. For example,  $u_{\zeta\zeta}$  is the second derivative of the function with respect to  $\zeta$ , and  $\sigma_{\zeta\zeta}$  is the  $ij$ -th member of the inverse of the matrix  $[-\frac{\partial^2 L(\nu, \zeta | \underline{x})}{\partial \nu \partial \zeta}]^{-1}$  by  $(\nu, \zeta)$ . Also,  $p_{\nu} = \frac{a-1}{\nu} - b$ ,  $p_{\zeta} = \frac{c-1}{\zeta} - d$ . Under Squared loss function, we have:

$$u(\nu, \zeta) = \nu; u_{\nu} = 1; u_{\nu\nu} = u_{\zeta} = u_{\zeta\zeta} = u_{\nu\zeta} = u_{\zeta\nu} = 0.$$

So the Bayes estimator for  $\nu$  is as follows:

$$\hat{\nu}_{LindleySE} = E(\nu | \underline{x}) = \hat{\nu} + \frac{1}{2} [2\hat{p}_{\nu} \hat{\sigma}_{\nu\nu} + 2\hat{p}_{\zeta} \hat{\sigma}_{\nu\zeta} + \hat{\sigma}_{\nu\nu} (\hat{L}_{\nu\zeta\zeta} \hat{\sigma}_{\zeta\zeta} + \hat{L}_{\zeta\nu\nu} \hat{\sigma}_{\zeta\nu})]$$

$$+\hat{L}_{\nu\zeta\nu}\hat{\sigma}_{\nu\zeta}+\hat{L}_{\nu\nu\nu}\hat{\sigma}_{\nu\nu})+\hat{\sigma}_{\zeta\nu}(\hat{L}_{\zeta\zeta\zeta}\hat{\sigma}_{\zeta\zeta}+\hat{L}_{\zeta\nu\zeta}\hat{\sigma}_{\zeta\nu}+\hat{L}_{\nu\zeta\zeta}\hat{\sigma}_{\nu\zeta}+\hat{L}_{\nu\nu\zeta}\hat{\sigma}_{\nu\nu})).$$

where,  $L_{\nu\zeta} = \frac{\partial^2 L(\nu, \zeta | \underline{x})}{\partial \nu \partial \zeta}$ . Also, the Bayes estimator for  $\zeta$  is as follows:

$$u(\nu, \zeta) = \zeta; u_\zeta = 1; u_{\zeta\zeta} = u_\nu = u_{\nu\nu} = u_{\nu\zeta} = u_{\zeta\nu} = 0,$$

$$\hat{\varsigma}_{Lindley_{SE}} = E(\varsigma | \underline{x}) = \hat{\varsigma} + \frac{1}{2}[2\hat{p}_\varsigma\hat{\sigma}_{\varsigma\varsigma} + 2\hat{p}_\nu\hat{\sigma}_{\nu\varsigma} + \hat{\sigma}_{\varsigma\varsigma}(\hat{L}_{\varsigma\varsigma\varsigma}\hat{\sigma}_{\varsigma\varsigma} + \hat{L}_{\zeta\nu\varsigma}\hat{\sigma}_{\zeta\nu}$$

$$+\hat{L}_{\nu\varsigma\varsigma}\hat{\sigma}_{\nu\varsigma} + \hat{L}_{\nu\nu\varsigma}\hat{\sigma}_{\nu\nu})] + \hat{\sigma}_{\nu\varsigma}(\hat{L}_{\nu\varsigma\varsigma}\hat{\sigma}_{\nu\varsigma} + \hat{L}_{\zeta\nu\nu}\hat{\sigma}_{\zeta\nu} + \hat{L}_{\nu\zeta\nu}\hat{\sigma}_{\nu\zeta} + \hat{L}_{\nu\nu\nu}\hat{\sigma}_{\nu\nu})).$$

Bayes estimators under Linex loss function can be obtained as:  $u(\nu, \zeta) = e^{-h\nu}$ ;  $u_\nu = -he^{-h\nu}$ ;  $u_{\nu\nu} = h^2e^{-h\nu}$ ;  $u_\zeta = u_{\zeta\zeta} = u_{\nu\zeta} = u_{\zeta\nu} = 0$ ,

$$\begin{aligned} \hat{\nu}_{Lindley_{Linex}} &= -\frac{1}{h} \ln E(e^{-h\nu} | \underline{x}) = -\frac{1}{h} \ln[e^{-h\hat{\nu}} + \frac{1}{2}(h^2e^{-h\hat{\nu}}\hat{\sigma}_{\nu\nu} \\ &- he^{-h\hat{\nu}}(2\hat{p}_\nu\hat{\sigma}_{\nu\nu} + 2\hat{p}_\varsigma\hat{\sigma}_{\nu\varsigma} + \hat{\sigma}_{\nu\nu}^2\hat{L}_{\nu\nu\nu} + \hat{\sigma}_{\nu\nu}\hat{\sigma}_{\varsigma\varsigma}\hat{L}_{\varsigma\varsigma\nu} \\ &+ 2\hat{\sigma}_{\nu\varsigma}\hat{\sigma}_{\varsigma\nu}\hat{L}_{\nu\varsigma\varsigma} + \hat{\sigma}_{\nu\varsigma}\hat{\sigma}_{\varsigma\varsigma}\hat{L}_{\varsigma\varsigma\varsigma}))]. \end{aligned}$$

Also, the Bayes estimator for  $\zeta$  is as follows:

$$u(\nu, \zeta) = e^{-h\zeta}; u_\zeta = -he^{-h\zeta}; u_{\zeta\zeta} = h^2e^{-h\zeta}; u_\nu = u_{\zeta\zeta} = u_{\nu\zeta} = u_{\zeta\nu} = 0,$$

$$u(\nu, \varsigma) = e^{-h\varsigma}; u_\varsigma = -he^{-h\varsigma}; u_{\varsigma\varsigma} = h^2e^{-h\varsigma}; u_\nu = u_{\varsigma\varsigma} = u_{\nu\varsigma} = u_{\varsigma\nu} = 0,$$

$$\begin{aligned} \hat{\nu}_{Lindley_{Linex}} &= -\frac{1}{h} \ln E(e^{-h\zeta} | \underline{x}) = -\frac{1}{h} \ln[e^{-h\hat{\varsigma}} + \frac{1}{2}(h^2e^{-h\hat{\varsigma}}\hat{\sigma}_{\varsigma\varsigma} \\ &- he^{-h\hat{\varsigma}}(2\hat{p}_\varsigma\hat{\sigma}_{\varsigma\varsigma} + 2\hat{p}_\nu\hat{\sigma}_{\nu\varsigma} + \hat{\sigma}_{\varsigma\varsigma}^2\hat{L}_{\varsigma\varsigma\varsigma} + 3\hat{\sigma}_{\nu\varsigma}\hat{\sigma}_{\varsigma\varsigma}\hat{L}_{\nu\varsigma\varsigma} + \hat{\sigma}_{\nu\nu}\hat{\sigma}_{\varsigma\varsigma}\hat{L}_{\nu\nu\nu}))]. \end{aligned}$$

Here,  $\hat{\sigma}_{ij}$  is the  $(i, j)$ -th element of the inverse of the matrix  $[-\frac{\partial^2 L(\nu, \zeta | \underline{x})}{\partial \nu \partial \zeta}]^{-1}|_{(\hat{\nu}, \hat{\zeta})}$ . By defining,  $\hat{\sigma}_{\nu\nu} = \frac{\Psi_2}{\Upsilon}$ ;  $\hat{\sigma}_{\nu\varsigma} = \frac{\Psi_3}{\Upsilon}$ ;  $\Upsilon = \Psi_1\Psi_2 - (\Psi_3)^2$ ,  $\hat{L}_{\nu\nu\nu} = \frac{\partial^3 L}{\partial \nu^3} = \frac{2D}{\nu^3}$ ,  $\hat{L}_{\nu\nu\zeta} = \hat{L}_{\nu\zeta\nu} = \hat{L}_{\zeta\nu\nu} = 0$ ,  $\Psi_1 = \frac{\partial^2 L}{\partial \nu^2}$ ,  $\Psi_2 = \frac{\partial^2 L}{\partial \nu^2}$ ,  $\Psi_3 = \frac{\partial^2 L}{\partial \nu \partial \zeta}$ .

The Bayes estimates of  $\nu$  under squared error (SE) and Linex loss functions are given by:

$$\begin{aligned} \hat{\nu}_{Lindley_{SE}} &= \hat{\nu} + \frac{1}{2} \left[ 2 \left( \frac{a-1}{\hat{\nu}} - b \right) \left( \frac{\Psi_2}{\Upsilon} \right) + 2 \left( \frac{c-1}{\hat{\varsigma}} - d \right) \left( \frac{\Psi_3}{\Upsilon} \right) + \left( \frac{\Psi_2}{\Upsilon} \right)^2 \hat{L}_{\nu\nu\nu} \right. \\ &\quad \left. + \left( \frac{\Psi_2\Psi_1}{\Upsilon^2} \right) \hat{L}_{\varsigma\varsigma\nu} + 2 \left( \frac{\Psi_3}{\Upsilon} \right)^2 \hat{L}_{\nu\varsigma\varsigma} + \left( \frac{\Psi_3\Psi_1}{\Upsilon^2} \right) \hat{L}_{\varsigma\varsigma\varsigma} \right], \end{aligned} \quad (14)$$

and

$$\hat{\nu}_{Lindley_{Linex}} = -\frac{1}{h} \ln \left\{ e^{-h\hat{\nu}} + \frac{1}{2} \left[ h^2 e^{-h\hat{\nu}} \left( \frac{\Psi_2}{\Upsilon} \right) - h e^{-h\hat{\nu}} \left( 2 \left( \frac{a-1}{\hat{\nu}} - b \right) \left( \frac{\Psi_2}{\Upsilon} \right) \right. \right. \right.$$

$$\begin{aligned}
& 2\left(\frac{c-1}{\hat{\zeta}} - d\right)\left(\frac{\Psi_3}{\Upsilon}\right) + \left(\frac{\Psi_2}{\Upsilon}\right)^2 \hat{L}_{\nu\nu\nu} \\
& + \left(\frac{\Psi_2\Psi_1}{\Upsilon^2}\right)\hat{L}_{\varsigma\varsigma\nu} + 2\left(\frac{\Psi_3}{\Upsilon}\right)^2 \hat{L}_{\nu\varsigma\varsigma} + \left(\frac{\Psi_3\Psi_1}{\Upsilon^2}\right)\hat{L}_{\varsigma\varsigma\varsigma} \Bigg] \Bigg\}, \quad (15)
\end{aligned}$$

Respectively, where,

$$\hat{L}_{\zeta\zeta\zeta} = \frac{\partial^3 L}{\partial \zeta^3} = \frac{2D}{\zeta^3} - \nu 2^{-\zeta} \left[ \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} \ln^3 \left( \frac{1}{2} w_{i:m:n} \right) + R^* w_T^{\zeta} \ln^3 \left( \frac{1}{2} w_T \right) \right],$$

and

$$\hat{L}_{\nu\zeta\zeta} = \hat{L}_{\zeta\nu\zeta} = \hat{L}_{\zeta\zeta\nu} = -2^{-\zeta} \left[ \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} \ln^2 \left( \frac{1}{2} w_{i:m:n} \right) + R^* w_T^{\zeta} \ln^2 \left( \frac{1}{2} w_T \right) \right].$$

Further, the Bayes estimates of  $\zeta$  with respect to the SE and Linex loss functions can be derived similarly.

### 3.2. Metropolis-Hastings (MH) algorithm

The MH algorithm ([11]- [17]) is a MCMC method used for sampling from a probability distribution that is difficult to sample directly. The algorithm involves proposing a new candidate sample, evaluating its acceptance probability, and accepting or rejecting the candidate based on that probability. The MH and Gibbs sampler methods found in Hastings [11] are among the several popular algorithms used for generating posterior samples. The MH method is popular because it does not need the full conditionals from the given posterior like Gibbs sampler. The Markov chain can be defined up to proportionality as a product of the likelihood function and the prior distribution. Let  $\theta = (\nu, \varsigma)$  denotes the vector valued parameter and  $Q(\theta, \tilde{\theta})$  presents a Markov Kernel, where  $\theta$  is the current realization and  $\tilde{\theta}$  is the next proposed realization from  $Q(\theta, \tilde{\theta})$ . In MH algorithm, we accept the candidate point  $\tilde{\theta}$  from the given posterior with the acceptance rate  $\min(1, \frac{\pi(\tilde{\theta}|data)Q(\theta, \tilde{\theta})}{\pi(\theta|data)Q(\theta, \tilde{\theta})})$ . Understanding the acceptance rate is crucial for the efficiency and convergence of the algorithm. The acceptance probability determines the probability of moving from the current state  $\theta$  to the proposed state  $\tilde{\theta}$ . This compares the probability of being in the proposed state  $\tilde{\theta}$  versus the current state  $\theta$ , adjusted by the proposal probabilities in both directions. This ratio ensures that the Markov chain satisfies the detailed balance condition, which is sufficient for the chain to have  $\pi(\theta|data)$  as its stationary distribution. Also, to choose the proposed density, the posterior should be re-parameterized so that all parameters are supported on the true line and the multivariate normal distribution is applied as the proposal distribution with mean vector  $\Theta$  and a Hessian based approximation to the variance covariance matrix [18]. The MH approach termed as Algorithm 1 with Gibbs technique is used to generate posterior samples, accordingly both point and credible interval estimates of parameters are computed in consequence.

#### Algorithm: The MCMC Technique

*Step 1:* Assign an initial value of parameters with their MLEs and set  $C = 1$ .

*Step 2:* Generate  $\nu$  using  $Gamma(D + a, b + 2^{-\zeta} \sum_{i=1}^D (R_i + 1) w_{i:m:n}^{\zeta} + R^* w_T^{\zeta})$ .

*Step 3:* Generate  $\zeta^*$  from the normal proposal distribution  $N(\zeta^{(C-1)}, \text{Variance}(\zeta))$ .

*Step 4:* Compute  $E_\zeta = \min \left( 1, \frac{\pi_2(\zeta^* | \nu^{(C-1)}, \text{data})}{\pi_2(\zeta^{(C-1)} | \nu^{(C-1)}, \text{data})} \right)$ .

*Step 5:* if  $\tau < E_\zeta$ , put  $\zeta^{(C)} = \zeta^*$ , else set  $\zeta^{(C)} = \zeta^{(C-1)}$ , where,  $\tau \sim \text{Uniform}(0, 1)$ .

*Step 6:* Set  $C = C + 1$ .

*Step 7:* Repeat steps 1-6,  $N$  times and gain  $\nu^{(1)}, \dots, \nu^{(N)}$  and  $\zeta^{(1)}, \dots, \zeta^{(N)}$ .

*Step 8:* The Bayesian estimates of  $\theta = (\nu, \zeta)$  under SE and Linex loss functions are:

$$\hat{\theta}_{Bayes_{SE}} = \frac{1}{N-b} \sum_{C=b+1}^N \theta^{(C)}$$

and

$$\hat{\theta}_{Bayes_{Linex}} = -\frac{1}{h} \ln \left[ \frac{1}{N-b} \sum_{C=b+1}^N e^{h\theta^{(C)}} \right] \quad (16)$$

$b$  is the burn-in period.

*Step 9:* Obtain highest posterior density credible interval as:

Generated MCMC samples of the parameters have been arranged in ascending order as  $(\nu^{(1)}, \dots, \nu^{(N)})$  and  $(\zeta^{(1)}, \dots, \zeta^{(N)})$ .

The HPD credible intervals of  $\theta$  with  $s$  significance level are given by

$$[\theta_{(Ns/2)}, \theta_{(N(1-s/2))}]. \quad (17)$$

### 3.3. Bayesian hyper-parameter selection

One of the main issues in the Bayesian approach is determining the hyper-parameters. Moreover, the values of the hyper-parameters for the unknown parameters are chosen based on two types of information: the mean expected value and the uncertainty of the unknown parameter in the model. Below are the steps to determine the hyper-parameters values.

Step 1: Determine the true value of  $\theta = (\nu, \zeta)$ .

Step 2: Simulate a random sample from WeHL model.

Step 3: Obtain the MLE of  $\theta$  as  $\hat{\theta}$ .

Step 4: Repeat Steps 2 and 3,  $N$  times to calculate  $(\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(N)})$ .

Step 5: Assign the prior mean and variance of the gamma density to the sample mean and sample variance of  $\hat{\theta}_{(i)}$  as:

$$E(\hat{\theta}^{(i)}) \rightarrow \frac{1}{N} \sum_{i=1}^N \hat{\theta}^{(i)} = \frac{a}{b}$$

and

$$\text{Var}(\hat{\theta}^{(i)}) \rightarrow \frac{1}{N-1} \sum_{i=1}^N \left( \hat{\theta}^{(i)} - \left( \frac{1}{N} \sum_{i=1}^N \hat{\theta}^{(i)} \right)^2 \right) = \frac{a}{b^2}.$$

Here,  $a$  and  $b$  are the parameters of the gamma prior distribution.

Step 6: The proposed hyper-parameters can be selected using the following formula:

$$a = \frac{\hat{\nu}^2}{\frac{1}{N-1} \sum_{i=1}^N (v_i - \hat{\nu})^2}, \quad b = \frac{\hat{\nu}}{\frac{1}{N-1} \sum_{i=1}^N (\nu_i - \hat{\nu})^2},$$

$$c = \frac{\hat{\zeta}^2}{\frac{1}{N-1} \sum_{i=1}^N (\zeta_i - \hat{\zeta})^2}, \quad d = \frac{\hat{\zeta}}{\frac{1}{N-1} \sum_{i=1}^N (\zeta_i - \hat{\zeta})^2}$$

## 4. Simulation design

In this section, a Monte Carlo simulation study is executed to compare the efficacy of methods outlined earlier. The comparison involves varying  $(n, m)$  :  $(40, 20), (40, 30), (60, 30), (60, 40)$ , and  $k$  and progressive censoring ( $R$ ). Additionally, for each set of  $(n, m)$ , two different  $T_i$ ;  $i = 1, 2$  values:  $T_1 = 1.5, T_2 = 2.5$  and  $T_1 = 2.5, T_2 = 3.5$  are utilized. The true parameter values are deemed as  $(\nu, \zeta) = (1.2, 0.9)$ . Based on the following considerations, the performance of the estimates has been compared.

**Mean squared error (MSE):**  $MSE = \frac{1}{N} \sum_{i=1}^N (\theta_i - \hat{\theta}_i)^2$ , where  $\theta_i$  represents parameters, whereas  $\hat{\theta}$  represents their estimates and  $N$  is the number of iterations. The greater performance of the estimations is indicated by the smaller value of MSE.

**Average width:** Average width of  $100(1 - s)\%$  interval estimates has been assessed. Better performance of the interval estimates is correlated with narrower width.

**Coverage probability:** The probability of containing the true values of the parameters in between the interval estimates.

To generate *UnPrHyCeSc*, the following algorithm has been constructed.

**Step 1:** Prefix  $n, m, k, T_1, T_2$  and  $R = (R_1, R_2, \dots, R_m)$ ;  $\sum_{i=1}^m R_i = n - m$ .

**Step 2:** Generate  $m$  independent observations  $g_1, g_2, \dots, g_m$  from standard Uniform distribution.

**Step 3:** Compute  $H_j = G_j^{(j + \sum_{i=m-j+1}^m R_i)^{-1}}$ ,  $j = 1, \dots, m$  using *ProgresCeSc*.

**Step 4:** Create  $Z_j = 1 - \prod_{j=m-i+1}^m H_j$ ;  $j = 1, \dots, m$  as a progressive sample.

**Step 5:** Generate progressive *WeHL* sample using

$$X_j = \ln \left[ \left( -\frac{2^\zeta \ln(1 - Z_j)}{\nu} \right)^{1/\zeta} + 1 \right]; \quad j = 1, \dots, m.$$

**Step 6:** Generate *UnPrHyCeSc* by using

$$T^* = \max\{\min\{X_{k:m:n}, T_2\}, \min\{X_{m:m:n}, T_1\}\}.$$

It is worth mentioning that we have used *R programming language* software with different useful packages. The functions that we applied for the modeling process are:

**Function** *"optim"* for *"stats"* package: we use this library to estimate the unknown

parameters of the *WeHL* distribution (for solving the non-linear equation).

**Function “solve”** from “*base R*”: we use this function to compute the inverse of this matrix, which is used to estimate the covariance matrix of the parameter estimates.

**Function “Deriv”** for “*Deriv*” package: we use this function to compute the required derivatives of the logarithm of the likelihood function.

**Function “fitdist”** for “*fitdistrplus*” package: it is used to fit a probability distribution to a given dataset.

**Function “gofstat”** for “*fitdistrplus*” package: we use this library in the application part for calculating the goodness-of-fit tests with its associated *p*-value.

**Function “MCMC”** for “*coda*” package: in the context of Bayesian analysis with Markov Chain Monte Carlo (MCMC) methods, we use this function to convert the sequence of samples from the posterior distribution of a parameter into an MCMC object.

**Function “HPDinterval”** for “*coda*” package: we use this function to compute the HPD intervals for Markov Chain Monte Carlo samples.

For each  $n, m, k, T_1$  and  $T_2$ , three *CeScs* are adopted as:

**Censoring Scheme I (Left Censoring):**  $R_1 = n - m$  and  $R_2 = R_3 = \dots = R_m = 0$ .

**Censoring Scheme II (Right Censoring):**  $R_1 = R_2 = \dots = R_{m-1} = 0$  and  $R_m = n - m$ .

**Censoring Scheme III (Middle Censoring):**  $R_1, \dots, R_{m-1} = 1$  and  $R_m = n - 2m + 1$ .

The information prior set up ( $a_1 = 1.2, b_1 = 1, a_2 = 1.8, b_2 = 2$ ) is applied for Bayesian computation.

Since MCMC algorithms start with initial parameter values that may be far from the true values or areas of high probability in the target distribution, the burn-in period refers to the initial set of iterations in an MCMC simulation that are discarded before collecting samples for analysis. This period allows the Markov chain to converge towards its stationary distribution, reducing the influence of the starting values on the final results.

To mitigate dependency on initial values, we employed the MCMC technique, conducting 10000 simulation repetitions and discarding the initial 1000 ( $b = 1000$ ) values as burn-in. Based on these set up assumptions, we show the numerical results in the Tables 1-6. The average point estimates (also, the MSE values of each estimates are placed inside the parenthesis.) of  $\nu$  and  $\zeta$  are presented in Tables 1-4 and also the interval estimates with 0.05 significance level are tabulated in Tables 5-6. The following observations are pointed out from the simulation study.

I. Decreasing pattern is observed in the values of the MSEs and biases with an increase in the values of  $n$  for *UnPrHyCeSc*.

II. With an increase in  $m$ , the MSEs and biases are decreasing for fixed values of  $n, k, T_1$  and  $T_2$  and this is also quite obvious due to the fact that increasing  $m$  may result in more failures.

III. For fixed values of  $n, k$ , and  $m$  with an increment in time  $T_i$ , MSEs, biases, and average interval lengths are results with decreasing values. Increasing  $T$  provides

more testing time to allow more failure to be observed.

IV. The Bayes estimates have smaller MSEs, especially in Markov Chain Monte Carlo method and also in Linex loss function.

V. In all cases, for fixed  $n, m$ , and  $T_i$ , the MSEs and average interval lengths of all parameters decrease as  $k$  increases.

VI. The average interval lengths are decreasing with larger sample sizes. Also, the HPD bounds are shorter than approximate confidence interval, as we would expect. Further, the *approximate confidence intervals* are better than HPD interval estimates in terms of coverage probabilities.

VII. The effect of the deletion position on the estimation performance has been observed. The theoretical findings based on the *CeSc II* indicate the more effective performance of this censoring than other proposed *CeScs*.

**Table 1**  
Average estimates and MSEs of  $\zeta$  for  $T_1 = 1.5$  and  $T_2 = 2.5$

| (n,m)   | k  | CS  | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|---------|----|-----|-----------------------|--------------------------|----------|----------|----------|
|         |    |     |                       | Lindley                  |          | MH       |          |
|         |    |     |                       | SE                       | LINEX    | SE       | LINEX    |
| (40,20) | 4  | I   | 1.1440                | 1.1238                   | 1.1275   | 1.1802   | 1.1336   |
|         |    |     | (0.3349)              | (0.2833)                 | (0.2785) | (0.2829) | (0.2596) |
|         |    | II  | 1.2123                | 1.2817                   | 1.1855   | 1.2035   | 1.1606   |
|         |    |     | (0.3029)              | (0.2782)                 | (0.2694) | (0.2421) | (0.2416) |
|         |    | III | 1.1996                | 1.2375                   | 1.2210   | 1.2566   | 1.1036   |
|         |    |     | (0.4788)              | (0.3967)                 | (0.3964) | (0.3592) | (0.3379) |
|         | 8  | I   | 1.2375                | 1.1175                   | 1.1388   | 1.2531   | 1.2575   |
|         |    |     | (0.3123)              | (0.2680)                 | (0.2616) | (0.2574) | (0.2387) |
|         |    | II  | 1.2189                | 1.2011                   | 1.2024   | 1.1533   | 1.1882   |
|         |    |     | (0.2994)              | (0.2684)                 | (0.2580) | (0.2386) | (0.2276) |
|         |    | III | 1.2372                | 1.1671                   | 1.1442   | 1.1966   | 1.2387   |
|         |    |     | (0.4436)              | (0.3213)                 | (0.3183) | (0.2972) | (0.2842) |
| (40,30) | 8  | I   | 1.1395                | 1.1307                   | 1.2580   | 1.2043   | 1.1415   |
|         |    |     | (0.2920)              | (0.2619)                 | (0.2590) | (0.2351) | (0.2312) |
|         |    | II  | 1.1508                | 1.1505                   | 1.1936   | 1.1013   | 1.2885   |
|         |    |     | (0.2729)              | (0.2595)                 | (0.2413) | (0.2335) | (0.2182) |
|         |    | III | 1.1919                | 1.2465                   | 1.1525   | 1.1804   | 1.1813   |
|         |    |     | (0.3948)              | (0.3878)                 | (0.3733) | (0.3569) | (0.3489) |
|         | 16 | I   | 1.2149                | 1.1329                   | 1.2695   | 1.1666   | 1.1660   |
|         |    |     | (0.2808)              | (0.2486)                 | (0.2296) | (0.2276) | (0.2192) |
|         |    | II  | 1.1973                | 1.2971                   | 1.2628   | 1.1768   | 1.2123   |
|         |    |     | (0.2515)              | (0.2198)                 | (0.1956) | (0.2163) | (0.1832) |
|         |    | III | 1.2223                | 1.1854                   | 1.1670   | 1.1412   | 1.1419   |
|         |    |     | (0.3852)              | (0.3669)                 | (0.3579) | (0.3522) | (0.2410) |
| (60,30) | 10 | I   | 1.1789                | 1.2049                   | 1.2424   | 1.1197   | 1.2110   |

Table 1 (continued)

| (n,m)   | k  | CS  | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|---------|----|-----|-----------------------|--------------------------|----------|----------|----------|
|         |    |     |                       | Lindley                  |          | MH       |          |
|         |    |     |                       | SE                       | LINEX    | SE       | LINEX    |
|         |    | II  | (0.2673)              | (0.2176)                 | (0.1995) | (0.1825) | (0.1820) |
|         |    |     | 1.1811                | 1.2054                   | 1.2914   | 1.1179   | 1.2263   |
|         |    | III | (0.2243)              | (0.1882)                 | (0.1176) | (0.0722) | (0.0514) |
|         |    |     | 1.2798                | 1.1468                   | 1.1056   | 1.1736   | 1.2746   |
|         |    | I   | (0.3690)              | (0.3465)                 | (0.3405) | (0.3392) | (0.3317) |
|         |    |     | 1.2499                | 1.2825                   | 1.1567   | 1.1188   | 1.2251   |
|         | 15 | II  | (0.2501)              | (0.1830)                 | (0.1798) | (0.1757) | (0.1704) |
|         |    |     | 1.2805                | 1.1596                   | 1.1801   | 1.1058   | 1.1217   |
|         |    | III | (0.1171)              | (0.0693)                 | (0.0525) | (0.0465) | (0.0384) |
|         |    |     | 1.1506                | 1.1321                   | 1.1805   | 1.1772   | 1.1975   |
|         |    |     | (0.3522)              | (0.3423)                 | (0.3398) | (0.3302) | (0.3278) |
|         |    |     |                       |                          |          |          |          |
| (60,40) | 15 | I   | 1.1043                | 1.2400                   | 1.1768   | 1.2620   | 1.2144   |
|         |    |     | (0.2311)              | (0.1560)                 | (0.1525) | (0.1518) | (0.1498) |
|         |    | II  | 1.1072                | 1.2481                   | 1.2769   | 1.2747   | 1.1803   |
|         |    |     | (0.1033)              | (0.0516)                 | (0.0478) | (0.0426) | (0.0283) |
|         |    | III | 1.2075                | 1.1377                   | 1.1012   | 1.2258   | 1.2594   |
|         |    |     | (0.3444)              | (0.3202)                 | (0.3167) | (0.3100) | (0.3116) |
|         | 20 | I   | 1.2205                | 1.1143                   | 1.1635   | 1.1139   | 1.1472   |
|         |    |     | (0.2291)              | (0.1502)                 | (0.1435) | (0.1429) | (0.1314) |
|         |    | II  | 1.2234                | 1.1239                   | 1.2792   | 1.1857   | 1.2001   |
|         |    |     | (0.0670)              | (0.0456)                 | (0.0379) | (0.0301) | (0.0258) |
|         |    | III | 1.2375                | 1.1585                   | 1.1148   | 1.1654   | 1.2711   |
|         |    |     | (0.2858)              | (0.2797)                 | (0.2491) | (0.2525) | (0.2187) |

Table 2

Average estimates and MSEs of  $\zeta$  for  $T_1 = 2.5$  and  $T_2 = 3.5$ .

| (n,m)   | k   | CS | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|---------|-----|----|-----------------------|--------------------------|----------|----------|----------|
|         |     |    |                       | Lindley                  |          | MH       |          |
|         |     |    |                       | SE                       | LINEX    | SE       | LINEX    |
| (40,20) | 4   | I  | 1.1026                | 1.2263                   | 1.1777   | 1.2222   | 1.2519   |
|         |     |    | (0.2971)              | (0.1952)                 | (0.1847) | (0.1814) | (0.1800) |
|         |     | II | 1.2296                | 1.2371                   | 1.2335   | 1.1447   | 1.1858   |
|         |     |    | (0.2709)              | (0.2594)                 | (0.2230) | (0.2307) | (0.2054) |
|         | III |    | 1.1619                | 1.2421                   | 1.2338   | 1.1124   | 1.2170   |
|         |     |    | (0.3553)              | (0.3240)                 | (0.3213) | (0.3073) | (0.3005) |
| 8       | I   |    | 1.1099                | 1.2525                   | 1.1339   | 1.1014   | 1.1909   |
|         |     |    | (0.2700)              | (0.1883)                 | (0.1791) | (0.1743) | (0.1726) |

Table 2 (continued)

| (n,m)   | k  | CS  | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|---------|----|-----|-----------------------|--------------------------|----------|----------|----------|
|         |    |     |                       | Lindley                  |          | MH       |          |
|         |    |     |                       | SE                       | LINEX    | SE       | LINEX    |
| (40,30) | 8  | II  | 1.1488                | 1.1858                   | 1.2363   | 1.1798   | 1.2159   |
|         |    |     | (0.2232)              | (0.1904)                 | (0.1541) | (0.1214) | (0.1078) |
|         |    | III | 1.1181                | 1.1045                   | 1.1452   | 1.2985   | 1.2871   |
|         |    |     | (0.3428)              | (0.3097)                 | (0.2999) | (0.2938) | (0.2898) |
|         |    | I   | 1.1592                | 1.1292                   | 1.1504   | 1.1019   | 1.2797   |
|         |    |     | (0.2462)              | (0.1311)                 | (0.1301) | (0.1229) | (0.1212) |
|         | 16 | II  | 1.2411                | 1.1605                   | 1.2649   | 1.1464   | 1.2503   |
|         |    |     | (0.1071)              | (0.0967)                 | (0.0877) | (0.0859) | (0.0545) |
|         |    | III | 1.1651                | 1.2733                   | 1.1944   | 1.2901   | 1.1820   |
|         |    |     | (0.3245)              | (0.3189)                 | (0.3123) | (0.3067) | (0.3001) |
|         |    | I   | 1.2214                | 1.1127                   | 1.2703   | 1.1101   | 1.1032   |
|         |    |     | (0.2231)              | (0.1291)                 | (0.1279) | (0.1208) | (0.1192) |
| (60,30) | 10 | II  | 1.2145                | 1.1031                   | 1.2574   | 1.1738   | 1.1737   |
|         |    |     | (0.0797)              | (0.0542)                 | (0.0472) | (0.0418) | (0.0374) |
|         |    | III | 1.1889                | 1.2331                   | 1.2734   | 1.2753   | 1.1681   |
|         |    |     | (0.3024)              | (0.2993)                 | (0.2859) | (0.2902) | (0.2780) |
|         |    | I   | 1.1799                | 1.2370                   | 1.1623   | 1.1403   | 1.2440   |
|         |    |     | (0.2087)              | (0.1188)                 | (0.1029) | (0.0998) | (0.0959) |
|         | 15 | II  | 1.2881                | 1.1004                   | 1.2396   | 1.2058   | 1.2377   |
|         |    |     | (0.0502)              | (0.0381)                 | (0.0236) | (0.0185) | (0.0180) |
|         |    | III | 1.1249                | 1.1757                   | 1.1056   | 1.1811   | 1.2488   |
|         |    |     | (0.2897)              | (0.2690)                 | (0.2680) | (0.2577) | (0.2346) |
|         |    | I   | 1.2452                | 1.2684                   | 1.2589   | 1.1029   | 1.1216   |
|         |    |     | (0.1916)              | (0.1027)                 | (0.0993) | (0.0975) | (0.0944) |
| (60,40) | 15 | II  | 1.2971                | 1.1673                   | 1.2299   | 1.1577   | 1.2392   |
|         |    |     | (0.0289)              | (0.0220)                 | (0.0184) | (0.0183) | (0.0166) |
|         |    | III | 1.1066                | 1.1941                   | 1.2542   | 1.1570   | 1.1228   |
|         |    |     | (0.2611)              | (0.2559)                 | (0.2493) | (0.2387) | (0.2341) |
|         |    | I   | 1.1808                | 1.1434                   | 1.1660   | 1.1302   | 1.2253   |
|         |    |     | (0.1786)              | (0.0940)                 | (0.0921) | (0.0915) | (0.0908) |
|         | 20 | II  | 1.2250                | 1.1365                   | 1.2871   | 1.2637   | 1.2808   |
|         |    |     | (0.0201)              | (0.0128)                 | (0.0662) | (0.0661) | (0.0643) |
|         |    | III | 1.1650                | 1.2559                   | 1.2320   | 1.1674   | 1.2791   |
|         |    |     | (0.2383)              | (0.2270)                 | (0.2184) | (0.2055) | (0.2025) |
|         |    | I   | 1.1734                | 1.2456                   | 1.2653   | 1.2782   | 1.1285   |
|         |    |     | (0.1489)              | (0.0915)                 | (0.0883) | (0.0876) | (0.0853) |
| (60,40) | 20 | II  | 1.2761                | 1.2334                   | 1.2417   | 1.2962   | 1.1501   |
|         |    |     | (0.0106)              | (0.0187)                 | (0.0129) | (0.0117) | (0.0125) |
|         |    | III | 1.1716                | 1.1160                   | 1.2813   | 1.1907   | 1.2592   |
|         |    |     | (0.2055)              | (0.1967)                 | (0.1911) | (0.1900) | (0.1837) |

**Table 3**  
Average estimates and MSEs of  $\nu$  for  $T_1 = 1.5$  and  $T_2 = 2.5$ .

| (n,m)   | k  | CeSc | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|---------|----|------|-----------------------|--------------------------|----------|----------|----------|
|         |    |      |                       | Lindley                  |          | MH       |          |
|         |    |      |                       | SE                       | LINEX    | SE       | LINEX    |
| (40,20) | 4  | I    | 0.7613                | 0.8434                   | 0.9264   | 0.8755   | 0.9401   |
|         |    |      | (0.0593)              | (0.0578)                 | (0.0570) | (0.0558) | (0.0545) |
|         |    | II   | 0.7876                | 0.9704                   | 0.9626   | 0.9868   | 0.8363   |
|         |    |      | (0.0363)              | (0.0289)                 | (0.0273) | (0.0246) | (0.0229) |
|         |    | III  | 0.7511                | 0.8775                   | 0.7634   | 0.8761   | 0.8033   |
|         |    |      | (0.0744)              | (0.0714)                 | (0.0703) | (0.0681) | (0.0672) |
|         | 8  | I    | 0.7988                | 0.8347                   | 0.8120   | 0.9094   | 0.8160   |
|         |    |      | (0.0479)              | (0.0421)                 | (0.0411) | (0.0395) | (0.0390) |
|         |    | II   | 0.7957                | 0.8478                   | 0.8835   | 0.8736   | 0.9668   |
|         |    |      | (0.0389)              | (0.0346)                 | (0.0380) | (0.0296) | (0.0285) |
|         |    | III  | 0.7519                | 0.8021                   | 0.8375   | 0.8591   | 0.9247   |
|         |    |      | (0.0729)              | (0.0698)                 | (0.0676) | (0.0678) | (0.0656) |
| (40,30) | 8  | I    | 0.7722                | 0.8315                   | 0.8795   | 0.8621   | 0.8796   |
|         |    |      | (0.0416)              | (0.0388)                 | (0.0378) | (0.0371) | (0.0359) |
|         |    | II   | 0.7831                | 0.9448                   | 0.9234   | 0.9122   | 0.8788   |
|         |    |      | (0.0298)              | (0.0282)                 | (0.0277) | (0.0268) | (0.0249) |
|         |    | III  | 0.7957                | 0.9144                   | 0.9157   | 0.8906   | 0.9572   |
|         |    |      | (0.0690)              | (0.0670)                 | (0.0656) | (0.0665) | (0.0648) |
|         | 16 | I    | 0.7545                | 0.9490                   | 0.9050   | 0.9397   | 0.9161   |
|         |    |      | (0.0406)              | (0.0371)                 | (0.0356) | (0.0363) | (0.0352) |
|         |    | II   | 0.7681                | 0.9576                   | 0.9415   | 0.9864   | 0.8682   |
|         |    |      | (0.0262)              | (0.0241)                 | (0.0234) | (0.0235) | (0.0227) |
|         |    | III  | 0.7593                | 0.9325                   | 0.9220   | 0.8506   | 0.8947   |
|         |    |      | (0.0676)              | (0.0637)                 | (0.0632) | (0.0615) | (0.0607) |
| (60,30) | 10 | I    | 0.7957                | 0.8324                   | 0.9624   | 0.9453   | 0.9251   |
|         |    |      | (0.0380)              | (0.0321)                 | (0.0302) | (0.0281) | (0.0274) |
|         |    | II   | 0.7994                | 0.9537                   | 0.9264   | 0.9861   | 0.9169   |
|         |    |      | (0.0237)              | (0.0229)                 | (0.0213) | (0.0200) | (0.0149) |
|         |    | III  | 0.7873                | 0.9440                   | 0.8676   | 0.8769   | 0.9506   |
|         |    |      | (0.0497)              | (0.0445)                 | (0.0433) | (0.0426) | (0.0422) |
|         | 15 | I    | 0.8270                | 0.9748                   | 0.9377   | 0.9570   | 0.9153   |
|         |    |      | (0.0375)              | (0.0319)                 | (0.0274) | (0.0252) | (0.0250) |
|         |    | II   | 0.7971                | 0.8621                   | 0.8957   | 0.9790   | 0.9356   |
|         |    |      | (0.0229)              | (0.0227)                 | (0.0211) | (0.0210) | (0.0208) |
|         |    | III  | 0.8997                | 0.9443                   | 0.9279   | 0.9345   | 0.9113   |
|         |    |      | (0.0474)              | (0.0424)                 | (0.0410) | (0.0394) | (0.0380) |
| (60,40) | 15 | I    | 0.7846                | 0.8581                   | 0.9725   | 0.8648   | 0.9293   |
|         |    | II   | 0.7934                | 0.8884                   | 0.9502   | 0.8976   | 0.8270   |

Table 3 (continued)

| (n,m) | k   | CeSc | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|-------|-----|------|-----------------------|--------------------------|----------|----------|----------|
|       |     |      |                       | Lindley                  |          | MH       |          |
|       |     |      |                       | SE                       | LINEX    | SE       | LINEX    |
| 20    | III |      | (0.0217)              | (0.0211)                 | (0.0208) | (0.0191) | (0.0177) |
|       |     |      | 0.8235                | 0.8414                   | 0.8636   | 0.9203   | 0.9366   |
|       |     |      | (0.0372)              | (0.0331)                 | (0.0320) | (0.0307) | (0.0299) |
|       | I   |      | 0.8546                | 0.9470                   | 0.9461   | 0.9197   | 0.9089   |
|       |     |      | (0.0340)              | (0.0295)                 | (0.0237) | (0.0212) | (0.0206) |
|       |     |      | 0.8655                | 0.9483                   | 0.8833   | 0.9647   | 0.8978   |
|       | II  |      | (0.0192)              | (0.0146)                 | (0.0135) | (0.0106) | (0.0100) |
|       |     |      | 0.8109                | 0.8864                   | 0.9677   | 0.8832   | 0.9333   |
|       |     |      | (0.0352)              | (0.0323)                 | (0.0303) | (0.0302) | (0.0299) |

Table 4

Average estimates and MSEs of  $\nu$  for  $T_1 = 2.5$  and  $T_2 = 3.5$ .

| (n,m)    | k  | CeSc | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|----------|----|------|-----------------------|--------------------------|----------|----------|----------|
|          |    |      |                       | Lindley                  |          | MH       |          |
|          |    |      |                       | SE                       | LINEX    | SE       | LINEX    |
| (40, 20) | 4  | I    | 0.8360                | 0.9752                   | 0.9385   | 0.9447   | 0.9263   |
|          |    |      | (0.0382)              | (0.0334)                 | (0.0328) | (0.0320) | (0.0316) |
|          |    |      | 0.9937                | 0.9713                   | 0.9514   | 0.9581   | 0.9332   |
|          |    | II   | (0.0299)              | (0.0274)                 | (0.0250) | (0.0256) | (0.0237) |
|          |    |      | 0.8899                | 0.8656                   | 0.8878   | 0.9695   | 0.9184   |
|          |    |      | (0.0418)              | (0.0363)                 | (0.0352) | (0.0333) | (0.0328) |
|          | 8  | I    | 0.9743                | 0.9514                   | 0.9241   | 0.8906   | 0.9033   |
|          |    |      | (0.0373)              | (0.0319)                 | (0.0308) | (0.0309) | (0.0282) |
|          |    |      | 0.8419                | 0.9197                   | 0.8525   | 0.8456   | 0.9022   |
|          |    | II   | (0.0290)              | (0.0266)                 | (0.0247) | (0.0226) | (0.0225) |
|          |    |      | 0.8537                | 0.8428                   | 0.9390   | 0.9315   | 0.8985   |
|          |    |      | (0.0397)              | (0.0331)                 | (0.0330) | (0.0319) | (0.0310) |
| (40, 30) | 8  | I    | 0.8246                | 0.9707                   | 0.8690   | 0.9593   | 0.9311   |
|          |    |      | (0.0348)              | (0.0295)                 | (0.0276) | (0.0290) | (0.0259) |
|          |    |      | 0.7897                | 0.9448                   | 0.9848   | 0.9283   | 0.9015   |
|          |    | II   | (0.0261)              | (0.0242)                 | (0.0232) | (0.0203) | (0.0175) |
|          |    |      | 0.8061                | 0.9966                   | 0.8987   | 0.9607   | 0.9314   |
|          |    |      | (0.0370)              | (0.0357)                 | (0.0344) | (0.0290) | (0.0278) |
|          | 16 | I    | 0.8510                | 0.9604                   | 0.8853   | 0.8832   | 0.9275   |
|          |    |      | (0.0342)              | (0.0282)                 | (0.0259) | (0.0279) | (0.0246) |
|          |    |      | 0.9744                | 0.9537                   | 0.9200   | 0.9409   | 0.8877   |
|          |    | II   | (0.0249)              | (0.0217)                 | (0.0206) | (0.0201) | (0.0171) |

Table 4 (continued)

| (n,m)    | k  | CeSc | ML Estimates<br>(MSE) | Bayes Estimates<br>(MSE) |          |          |          |
|----------|----|------|-----------------------|--------------------------|----------|----------|----------|
|          |    |      |                       | Lindley                  |          | MH       |          |
|          |    |      |                       | SE                       | LINEX    | SE       | LINEX    |
| (60, 30) | 10 | III  | 0.8657                | 0.9646                   | 0.8760   | 0.8908   | 0.9113   |
|          |    |      | (0.0359)              | (0.0296)                 | (0.0276) | (0.0280) | (0.0257) |
|          |    | I    | 0.8700                | 0.9567                   | 0.9548   | 0.9780   | 0.8681   |
|          |    |      | (0.0314)              | (0.0264)                 | (0.0232) | (0.0256) | (0.0219) |
|          |    | II   | 0.8715                | 0.9688                   | 0.8890   | 0.9408   | 0.9110   |
|          |    |      | (0.0199)              | (0.0171)                 | (0.0147) | (0.0142) | (0.0131) |
|          | 15 | III  | 0.8990                | 0.8582                   | 0.9215   | 0.9351   | 0.9037   |
|          |    |      | (0.0338)              | (0.0282)                 | (0.0268) | (0.0279) | (0.0246) |
|          |    | I    | 0.9741                | 0.9682                   | 0.9695   | 0.9644   | 0.9088   |
|          |    |      | (0.0307)              | (0.0251)                 | (0.0226) | (0.0249) | (0.0218) |
|          |    | II   | 0.8726                | 0.9894                   | 0.8593   | 0.9677   | 0.9189   |
|          |    |      | (0.0188)              | (0.0160)                 | (0.0136) | (0.0131) | (0.0119) |
|          | 20 | III  | 0.9729                | 0.9449                   | 0.9237   | 0.9134   | 0.9378   |
|          |    |      | (0.0322)              | (0.0279)                 | (0.0252) | (0.0243) | (0.0223) |
| (60, 40) | 15 | I    | 0.9772                | 0.8600                   | 0.8723   | 0.9154   | 0.8945   |
|          |    |      | (0.0277)              | (0.0237)                 | (0.0214) | (0.0222) | (0.0196) |
|          |    | II   | 0.9828                | 0.8663                   | 0.8784   | 0.8521   | 0.9185   |
|          |    |      | (0.0158)              | (0.0137)                 | (0.0127) | (0.0121) | (0.0107) |
|          |    | III  | 0.9989                | 0.9584                   | 0.8888   | 0.9372   | 0.9146   |
|          |    |      | (0.0281)              | (0.0267)                 | (0.0247) | (0.0219) | (0.0204) |
|          | 20 | I    | 0.9612                | 0.9433                   | 0.9528   | 0.8819   | 0.9274   |
|          |    |      | (0.0260)              | (0.0204)                 | (0.0176) | (0.0197) | (0.0171) |
|          |    | II   | 0.9794                | 0.9607                   | 0.9314   | 0.9400   | 0.8910   |
|          |    |      | (0.0121)              | (0.0108)                 | (0.0099) | (0.0077) | (0.0061) |
|          |    | III  | 0.9579                | 0.8664                   | 0.9173   | 0.9169   | 0.9051   |
|          |    |      | (0.0274)              | (0.0246)                 | (0.0226) | (0.0211) | (0.0204) |

**Table 5**

Average approximate confidence intervals (with coverage probability) for different censoring schemes

| (n,m)   | k  | CeSc | $T_1 = 1.5, T_2 = 2.5$    | $T_1 = 1.5, T_2 = 2.5$    | $T_1 = 2.5, T_2 = 3.5$    | $T_1 = 2.5, T_2 = 3.5$    |
|---------|----|------|---------------------------|---------------------------|---------------------------|---------------------------|
|         |    |      | $\zeta$                   | $\nu$                     | $\zeta$                   | $\nu$                     |
| (40,20) | 4  | I    | [0.4231, 1.9596]<br>0.934 | [0.5273, 1.8942]<br>0.951 | [0.8436, 1.8094]<br>0.938 | [0.5200, 1.2794]<br>0.952 |
|         |    | II   | [0.8873, 1.5035]<br>0.940 | [0.6983, 1.2767]<br>0.931 | [1.0307, 1.6372]<br>0.939 | [0.7549, 1.3159]<br>0.939 |
|         |    | III  | [0.8525, 1.6625]<br>0.943 | [0.6604, 1.3195]<br>0.927 | [0.9361, 1.5825]<br>0.943 | [0.5778, 1.2220]<br>0.961 |
|         | 8  | I    | [0.9975, 1.6010]<br>0.961 | [0.7372, 1.2280]<br>0.936 | [0.9340, 1.5079]<br>0.947 | [0.6305, 1.0794]<br>0.970 |
|         |    | II   | [1.0673, 1.5126]<br>0.944 | [0.7096, 1.0549]<br>0.938 | [0.9816, 1.4157]<br>0.941 | [0.8060, 1.2012]<br>0.962 |
|         |    | III  | [0.9526, 1.4548]<br>0.957 | [0.6784, 1.2528]<br>0.960 | [0.9266, 1.3983]<br>0.959 | [0.8274, 1.1894]<br>0.940 |
|         | 16 | I    | [1.1674, 1.5903]<br>0.937 | [0.6861, 1.2007]<br>0.936 | [0.7916, 1.2030]<br>0.940 | [0.7448, 1.0559]<br>0.942 |
|         |    | II   | [1.0879, 1.4574]<br>0.936 | [0.6959, 1.0888]<br>0.965 | [1.1447, 1.5111]<br>0.939 | [0.7323, 0.9205]<br>0.960 |
|         |    | III  | [0.8817, 1.2799]<br>0.943 | [0.7109, 1.1120]<br>0.939 | [1.1868, 1.5731]<br>0.940 | [0.6939, 0.9710]<br>0.963 |
|         |    | I    | [1.1274, 1.4916]<br>0.960 | [0.7383, 1.0050]<br>0.940 | [0.9427, 1.3012]<br>0.942 | [0.8400, 1.0865]<br>0.959 |
|         |    | II   | [0.9960, 1.3439]<br>0.940 | [0.7435, 1.0101]<br>0.938 | [1.1404, 1.4870]<br>0.957 | [0.8199, 1.1711]<br>0.970 |
|         |    | III  | [0.9920, 1.3479]<br>0.959 | [0.7392, 1.0462]<br>0.944 | [1.0712, 1.4228]<br>0.956 | [0.7351, 0.9748]<br>0.944 |
| (60,30) | 10 | I    | [0.8710, 1.2151]<br>0.941 | [0.7706, 1.0062]<br>0.939 | [1.0223, 1.3653]<br>0.945 | [0.7932, 1.0000]<br>0.940 |
|         |    | II   | [1.0319, 1.3680]<br>0.940 | [0.7894, 1.0010]<br>0.937 | [1.0113, 1.3472]<br>0.960 | [0.8913, 1.0925]<br>0.935 |
|         |    | III  | [1.0367, 1.3763]<br>0.936 | [0.7175, 0.9216]<br>0.941 | [1.0809, 1.4183]<br>0.942 | [0.7980, 1.0012]<br>0.970 |
|         | 15 | I    | [1.1534, 1.4865]<br>0.956 | [0.8058, 1.0239]<br>0.960 | [1.1375, 1.4570]<br>0.941 | [0.8048, 0.9951]<br>0.962 |
|         |    | II   | [1.0973, 1.4176]<br>0.942 | [0.7192, 0.9156]<br>0.940 | [1.2462, 1.3737]<br>0.960 | [0.8312, 1.8245]<br>0.941 |
|         |    | III  | [0.9123, 1.2402]<br>0.944 | [0.8972, 1.0848]<br>0.939 | [1.0328, 1.2112]<br>0.959 | [0.8568, 1.0332]<br>0.960 |
|         | 20 | I    | [0.9205, 1.2394]<br>0.952 | [0.7198, 0.9301]<br>0.942 | [1.0506, 1.3499]<br>0.952 | [0.8036, 0.9584]<br>0.961 |
|         |    | II   | [0.9405, 1.2348]<br>0.951 | [0.7301, 0.9291]<br>0.941 | [1.0804, 1.3718]<br>0.946 | [0.8631, 1.2043]<br>0.940 |
|         |    | III  | [1.1002, 1.3967]          | [0.7877, 0.9408]          | [1.1052, 1.3996]          | [0.8564, 1.9458]          |

|    |     |                  |                  |                  |                  |
|----|-----|------------------|------------------|------------------|------------------|
|    |     | 0.947            | 0.960            | 0.953            | 0.943            |
| 20 | I   | [1.1781, 1.4693] | [0.8072, 0.9952] | [1.0416, 1.3304] | [0.8440, 0.9972] |
|    |     | 0.950            | 0.942            | 0.949            | 0.958            |
|    | II  | [1.1472, 1.4177] | [0.8387, 0.9668] | [1.1444, 1.4126] | [0.8725, 0.9802] |
|    |     | 0.951            | 0.958            | 0.952            | 0.945            |
|    | III | [1.1178, 1.3993] | [0.7556, 0.9456] | [1.0443, 1.3204] | [0.8647, 0.9939] |
|    |     | 0.948            | 0.945            | 0.950            | 0.955            |

**Table 6**

Average credible intervals (with coverage probability) for different censoring schemes.

| (n,m)   | k       | CeSc | $T_1 = 1.5, T_2 = 2.5$ | $T_1 = 1.5, T_2 = 2.5$ | $T_1 = 2.5, T_2 = 3.5$ | $T_1 = 2.5, T_2 = 3.5$ |
|---------|---------|------|------------------------|------------------------|------------------------|------------------------|
|         |         |      | $\zeta$                | $\nu$                  | $\zeta$                | $\nu$                  |
| (40,20) | 4       | I    | [1.0464, 1.3085]       | [0.8298, 0.9539]       | [1.0624, 1.3242]       | [0.8383, 0.9614]       |
|         |         |      | 0.972                  | 0.973                  | 0.970                  | 0.970                  |
|         |         | II   | [1.1306, 1.3893]       | [0.8999, 1.1107]       | [0.9838, 1.2368]       | [0.8412, 0.9589]       |
|         | 8       | I    | [1.1294, 1.3905]       | [0.8692, 0.9922]       | [1.0544, 1.4048]       | [0.8309, 1.1119]       |
|         |         |      | 0.977                  | 0.982                  | 0.978                  | 0.980                  |
|         |         | II   | [1.0975, 1.3488]       | [0.8751, 0.9923]       | [1.0287, 1.2792]       | [0.8194, 0.9350]       |
|         | 16      | I    | [1.0947, 1.2940]       | [0.8306, 0.9361]       | [1.0358, 1.2834]       | [0.7473, 0.9526]       |
|         |         |      | 0.971                  | 0.968                  | 0.968                  | 0.968                  |
|         |         | III  | [1.0600, 1.3099]       | [0.8006, 0.9151]       | [1.1729, 1.4223]       | [0.8486, 0.9542]       |
| (40,30) | 8       | I    | [1.0699, 1.3154]       | [0.8487, 0.9512]       | [1.0472, 1.3577]       | [0.8994, 1.1498]       |
|         |         |      | 0.970                  | 0.980                  | 0.972                  | 0.970                  |
|         |         | II   | [1.0452, 1.2804]       | [0.8630, 0.9591]       | [1.0833, 1.3121]       | [0.8534, 0.9465]       |
|         |         |      | 0.931                  | 0.973                  | 0.970                  | 0.969                  |
|         |         | III  | [1.0059, 1.2440]       | [0.8582, 0.9586]       | [1.1481, 1.3853]       | [0.8642, 0.9609]       |
|         |         |      | 0.970                  | 0.971                  | 0.969                  | 0.973                  |
|         | 16      | I    | [1.0865, 1.3134]       | [0.9025, 1.1679]       | [1.0881, 1.3137]       | [0.8240, 0.9084]       |
|         |         |      | 0.933                  | 0.965                  | 0.970                  | 0.961                  |
|         |         | II   | [1.0807, 1.3012]       | [0.9291, 1.0302]       | [0.9967, 1.2155]       | [0.8994, 0.9776]       |
|         |         |      | 0.968                  | 0.969                  | 0.966                  | 0.969                  |
|         |         | III  | [1.0950, 1.3199]       | [0.8401, 0.9229]       | [1.1645, 1.3854]       | [0.8821, 0.9637]       |
|         |         |      | 0.969                  | 0.970                  | 0.970                  | 0.965                  |
|         | (60,30) | 10   | [1.0633, 1.2799]       | [0.8882, 0.9733]       | [1.1223, 1.3376]       | [0.9427, 1.0202]       |
|         |         |      | 0.975                  | 0.966                  | 0.941                  | 0.967                  |
|         |         | II   | [1.0704, 1.2747]       | [0.9007, 1.1377]       | [1.0888, 1.2894]       | [0.9031, 0.9715]       |
|         |         |      | 0.973                  | 0.964                  | 0.932                  | 0.960                  |
|         |         | III  | [1.1220, 1.3370]       | [0.8642, 0.9371]       | [1.1510, 1.3632]       | [0.8867, 0.9582]       |
|         |         |      | 0.969                  | 0.971                  | 0.960                  | 0.966                  |
| (60,30) | 15      | I    | [1.0999, 1.3004]       | [0.9090, 1.1269]       | [1.0438, 1.2608]       | [0.9118, 0.9781]       |
|         |         | II   | [1.0622, 1.1759]       | [0.9572, 1.0224]       | [1.0917, 1.1950]       | [0.8982, 1.7924]       |

|         |    |                  |                  |                  |                  |                  |
|---------|----|------------------|------------------|------------------|------------------|------------------|
|         |    |                  | 0.976            | 0.967            | 0.975            | 0.969            |
|         |    | III              | [1.1219, 1.3183] | [0.8657, 0.9510] | [1.0906, 1.2868] | [0.8794, 0.9846] |
| (60,40) | 15 |                  | 0.930            | 0.964            | 0.971            | 0.960            |
|         |    | I                | [1.0456, 1.2343] | [0.8614, 0.9255] | [1.1070, 1.3128] | [0.8687, 0.9312] |
|         |    |                  | 0.967            | 0.972            | 0.969            | 0.973            |
|         |    | II               | [1.1972, 1.3753] | [0.8640, 0.9250] | [1.1585, 1.3357] | [0.8124, 0.9932] |
|         |    |                  | 0.935            | 0.960            | 0.963            | 0.959            |
|         |    | III              | [1.1699, 1.3500] | [0.8702, 0.9325] | [1.1228, 1.3019] | [0.8897, 0.9809] |
|         |    |                  | 0.965            | 0.961            | 0.962            | 0.957            |
| 20      | I  | [1.1008, 1.2776] | [0.8650, 0.9236] | [1.1793, 1.3526] | [0.8596, 0.9266] |                  |
|         |    | 0.964            | 0.960            | 0.962            | 0.942            |                  |
|         |    | II               | [1.1524, 1.3187] | [0.9131, 0.9894] | [1.1905, 1.3562] | [0.8946, 0.9853] |
|         |    |                  | 0.962            | 0.959            | 0.964            | 0.957            |
|         |    | III              | [1.0912, 1.2412] | [0.8784, 0.9342] | [1.1234, 1.2891] | [0.8857, 0.9389] |
|         |    |                  | 0.940            | 0.960            | 0.960            | 0.955            |

## 5. Real Data Analysis

In this section, we apply our theoretical findings to real data. The original data are reported by Asadi et al. [2]. For a single cough, we consider the virus-containing micro drops injection velocity of 32 m/s (Figure 3). The data has been represented in Table 7. First, we used the Kolmogorov-Smirnov (**KS**) test to see if the data fit the *WeHL*, so that the corresponding P-values from the **KS** test is 0.9936. The *WeHL* is a reasonably good fit for this data. Figure 4 sketched the estimated density, empirical CDF, and P-P plot of our *WeHL* model. This Figure confirms that the distribution function of the *WeHL* is a good fit. The goodness-of-fit tests are conducted using the “*fitdistrplus*” package, which is designed specifically for use within the *R* statistical software. This package plays a crucial role in facilitating the comprehensive analysis, underscoring the technical foundation upon which our findings are based. Here we apply the “*Nelder-Mead*” as an optimization method to evaluate the unknown parameters of each distribution using maximum likelihood technique.

We adopt various schemes for the purpose of computation. Here, we consider four schemes as:

Scheme I:  $n = 19, m = 14, k = 10, T_1 = 1.32, T_2 = 1.35, R = (0^{*13}, 5)$ .

Scheme II:  $n = 19, m = 16, k = 10, T_1 = 1.30, T_2 = 1.36, R = (3, 0^{*15})$ .

Scheme III:  $n = 19, m = 16, k = 13, T_1 = 1.26, T_2 = 1.36, R = (0^{*13}, 5)$ .

Scheme IV:  $n = 19, m = 16, k = 14, T_1 = 1.25, T_2 = 1.28, R = (0^{*6}, 1^{*3}, 0^{*7})$ .

We perform the simulation algorithm for the Markov chain Monte Carlo method given in Subsection 3.2 with the iteration number  $D = 14000$ . Because we have no prior knowledge about the unknown parameters, we compute Bayes estimates with hyperparameters 0.0001.

Based on the above-mentioned dataset, MCMC samples for model parameters are plotted in Figure 5. Figure 5 shows the acceptable value for the convergence

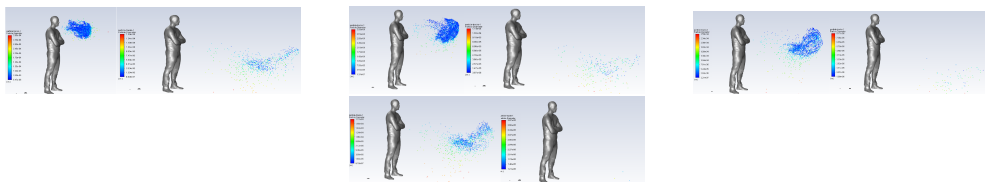


**Figure 3.** The 3D view of CFD simulation results.

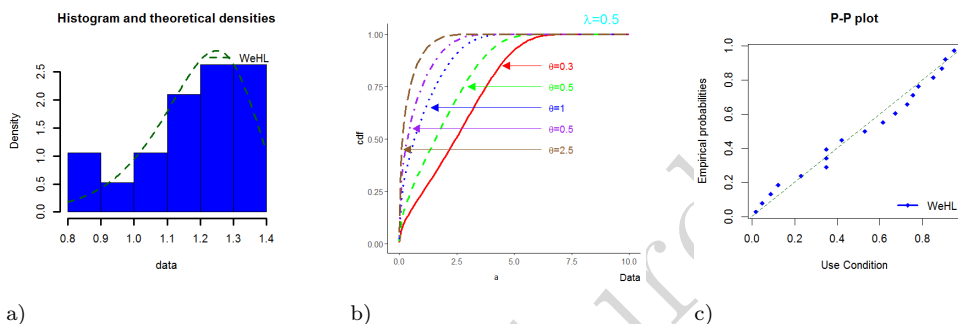


**Figure 4.** The plots of the (a) histogram (b) empirical CDF and (c) P-P plot under Virus-MD data.

of Markov chain Monte Carlo chains. In the scope of our study, the estimations of the unknown parameters are presented in Table 8 based on the different *CeSc* given above. According to Table 8, the Bayesian estimates were close to the ML estimates in all *CeSc*. Also, the average lengths (ALs) of the HPD intervals for all schemes were smaller than the approximate confidence intervals.

**Table 7**

The virus-containing micro drop data.

|                                                                                                                 |
|-----------------------------------------------------------------------------------------------------------------|
| 0.8, 0.9, 1.15, 1.15, 1.25, 1.3, 1.4, 0.97, 1.01, 1.09, 1.15, 1.18, 1.22,<br>1.27, 1.29, 1.31, 1.34, 1.36, 1.37 |
|-----------------------------------------------------------------------------------------------------------------|

## 6. Conclusion

In industrial life tests and reliability analysis, it is important to choose the appropriate *CeSc* to control the duration of the experiment time. In this article, we studied the inference for the *WeHL* distribution based on *UnPrHyCeSc*. Classical estimation and Bayesian approaches are used to estimate the unknown parameters. In the classical paradigm, the MLEs and the approximate confidence intervals of the parameters are computed. Two Bayesian approximation techniques have been utilized to calculate the Bayes and HPD interval estimations. The simulation investigations

**Table 8**  
Different estimates of  $\nu$  and  $\zeta$  for virus-containing micro drop.

| CeSc       | Parameters | MLE    | Lindley | MH     | Approximate<br>confidence<br>interval | HPD    |
|------------|------------|--------|---------|--------|---------------------------------------|--------|
| Scheme I   | $\nu$      | 0.3654 | 0.3521  | 0.3462 | 0.4235                                | 0.4015 |
|            | $\zeta$    | 5.9943 | 5.9243  | 5.8942 | 1.4235                                | 1.4013 |
| Scheme II  | $\nu$      | 0.3459 | 0.3275  | 0.3155 | 0.3995                                | 0.3903 |
|            | $\zeta$    | 5.8793 | 5.8701  | 5.8533 | 1.3994                                | 1.3885 |
| Scheme III | $\nu$      | 0.3649 | 0.3599  | 0.3096 | 0.3784                                | 0.3599 |
|            | $\zeta$    | 5.8577 | 5.8631  | 5.8421 | 1.3853                                | 1.3701 |
| Scheme IV  | $\nu$      | 0.3198 | 0.2998  | 0.2902 | 0.3575                                | 0.3525 |
|            | $\zeta$    | 5.6982 | 5.6611  | 5.6547 | 1.3810                                | 1.3768 |

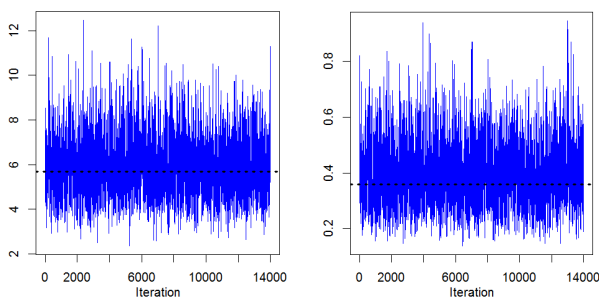


**Figure 5.** Trace plots of the posterior distribution of  $\nu$  and  $\zeta$  from scheme II.

demonstrated that the Bayes setup outperformed the maximum likelihood strategy pretty effectively. Further, the implementation of the proposed model is made by using the virus-containing micro drop data as an engineering application of the suggested approaches. With appropriate adjustments, this work may be extended to alternative censoring methods or the stress-strength and accelerated life test models.

## Appendix

The R programming code for the generation of the unified progressive hybrid censoring and for the computation of the MLEs and confidence intervals have been provided below:

```
rm(list=ls(all=TRUE))
#load required libraries
library(Deriv)
library(MASS)
library(nleqslv)
library(matlib)
library(coda)

# Define constants and parameters
n1 = 60; r1 = 30; k1 = 20
T1 = 1.5; T2 = 2.5
a = 1.2; b = 0.9
theta = c(a, b)
```

```

# Data generation function
generate_data <- function() {
  u1 = runif(n1, min = 0, max = log(2))
  x1 = log(1 + ((-(2^a) / b) * log(1 - u1))^(1 / a)))
  x1 = sort(x1)

  n1 = length(x1)
  d1 = 0; c1 = 0

  if (x1[r1] < T1) {
    d1 = sum(x1 < T1)
    c1 = T1
  }
  if ((x1[k1] < T1 & x1[r1] < T2) \mid (T1 < x1[k1] &
    x1[r1] < T2)) {
    d1 = r1
    c1 = x1[r1]
  }
  if ((x1[k1] < T1 & T2 < x1[r1]) \mid (T1 < x1[k1] &
    x1[k1] < T2 & T2 < x1[r1])) {
    d1 = sum(x1 < T2)
    c1 = T2
  }
  if (T2 < x1[k1]) {
    d1 = k1
    c1 = x1[k1]
  }

  x_1 = x1[1:d1]
  R = c(n1 - d1, rep(0, d1 - 1))
  Rs = n1 - d1 - sum(R)

  list(x_1 = x_1, R = R, Rs = Rs, c1 = c1, d1 = d1)
}

# Log-likelihood function
L1 = function(theta) {
  a = theta[1]
  b = theta[2]

  data = generate_data()
  x = data$x_1
  R = data$R
  Rs = data$Rs
  c = data$c1
  d = data$d1

  aal = (d * log(a)) + (d * log(b)) - (a * d * log(2)) +
    sum(x) + ((a - 1) * sum(log(exp(x) - 1))) -
    ((b * (2^a)) * (sum((R + 1) * ((exp(x) - 1)^a)) +
    (Rs * ((exp(c) - 1)^a))))
  return(-aal)
}

# Gradient of L1
dL1 = Deriv(L1, theta)

# Optimization setup
fit_l = optim(par = theta, L1, method = "L-BFGS-B",
  lower = c(0, 0),
  upper = c(Inf, Inf), hessian = TRUE)
fit_l_par = fit_l$par

# Safe optimization loop

```

```

safe_optim <- function(iteration) {
  tryCatch({
    fit <- optim(par = theta, fn = L1,
      method = "L-BFGS-B", lower = c(0, 0),
      upper = c(Inf, Inf), hessian = TRUE)
    if (is.null(fit$par) | mid any(is.na(fit$par))) {
      warning(paste("Iteration", iteration,
        "produced NA values."))
      return(rep(NA, length(theta)))
    }
    return(fit$par)
  }, error = function(e) {
    return(rep(NA, length(theta)))
  })
}

# Run optimization loop
N <- 10000
results <- t(sapply(1:N, safe_optim))
t <- results[complete.cases(results), ]
t1 = t[1, ]
t2 = t[2, ]

# Results and confidence intervals
alpha_mean = mean(t1)
beta_mean = mean(t2)
MSE_alpha = mean((t1 - a)^2)
MSE_beta = mean((t2 - b)^2)
fisher_info = solve(-fit_l$hessian)
prop_sigma = sqrt(diag(abs(fisher_info)))
lower = fit_l_par - 1.96 * prop_sigma
upper = fit_l_par + 1.96 * prop_sigma
interval1 = cbind(lower, upper)

# Metropolis-Hastings sampling function
MH <- function() {
  f1 = function(alpha, x) {
    J = (alpha^(d + b1 - 1)) * (2^(-d * alpha)) *
      (prod(exp(x) * (exp(x) - 1)^(alpha - 1))) *
      exp(-b2 * alpha) * (((b4 + ((2^(a)) *
      (sum((R + 1) * ((exp(x) - 1)^a)) +
      (Rs * ((exp(c) - 1)^a))))))^(-(d + b3)))
    return(J)
  }

  M1 = 14000
  burn = 500
  sigma = 0.9
  alphas1 = beta1 = rep(NA, (M1 + burn))
  alphas1[1] = 1
  u = runif(M1 + burn)
  k = 0

  for (l in 2:(M1 + burn)) {
    alphas1 = alphas1[l - 1]
    y = rnorm(1, alphas1, sigma)
    if (!is.na(f1(y, x)) && !is.na(f1(alphas1, x)) &&
      u[l] <= f1(y, x) * dnorm(alphas1, y, sigma) /
      (f1(alphas1, x) * dnorm(y, alphas1, sigma))) {
      alphas1[l] = y
    } else {
      alphas1[l] = alphas1
      k = k + 1
    }
  }
}

```

```

    beta1[l] = rgamma(1, (d + b3), (b4 + ((2^(a)) *
      (sum((R + 1) * ((exp(x) - 1)^a)) +
      (Rs * ((exp(c) - 1)^a))))))
  }

# Removing burn-in period
alpha1 = alpha1[(burn + 1):(burn + M1)]
beta1 = beta1[(burn + 1):(burn + M1)]

list(alpha1 = alpha1, beta1 = beta1)
}

# Run Metropolis-Hastings
result = MH()

# Trace plots
par(mfrow = c(1, 2))
plot(result$alpha1, type = 'l', col = 'blue', xlab = 'Iteration',
ylab = expression(zeta), main = "Trace plot for zeta")

```

## References

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