

Paweł J. Mitkowski*, Wojciech Mitkowski*

Stepped Basic Function in the Asymptotic Homogenization of an Elliptic System

1. Introduction

The common feature of various homogenization methods [2, p. 8, 16 and further] is that the mathematical model is constructed by carrying the information available in micro scale to the macro scale level. In the asymptotic homogenization, the periodicity of the structure, is the fundamental assumption [1, 3, 4, 5, 6]. In the one-dimensional case the increase of number of heterogeneities in the sample of length d (the macroscopic dimension in the system of variable x) is realized by successive reduction of basic cell (basic function – see Fig. 1 at $n = 0$) using homothetic transformation (preserving the majority of structural qualities of the medium, e.g. the proportions between the “inclusions”) of scale $\varepsilon = l/d$, where l is a dimension (macroscopic) of a single periodic cell. In Figure 1 the process of reduction is shown for $n = 0$ and $n = 1$, and $\varepsilon = 1/2$.

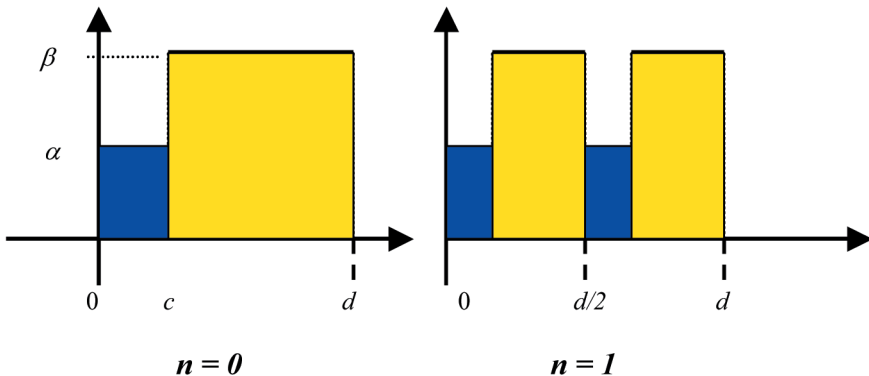


Fig. 1. Process of reduction for $n = 0$ and $n = 1$

* AGH University of Science and Technology, Krakow, Poland

The asymptotic homogenization consists in searching of macroscopic description, of considered process, in the form of suitable limits at $\varepsilon \rightarrow 0$. For example we study the following elliptic equation (we use a simplified notation from the monograph [6, p. 29, 45])

$$-\frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = f(x), \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega$$
(1)

or equivalently

$$-\frac{\partial p_i}{\partial x_i} = f, \quad p_i = a_{ij}(x) \frac{\partial u}{\partial x_j}$$
(2)

where $\Omega \subset R^n$ is a bounded domain of the space R^n of coordinates x_i with regular boundary $\partial\Omega$. Depending on the interpretation of the elliptic equation (1) the variable p is the heat flux, electric displacement or magnetic induction. The symmetric matrix $[a_{ij}(x)]$ describes a physical property of a material. If the material is homogeneous, then a_{ij} does not depend on x , if it is not homogeneous a_{ij} effectively depends on x . For materials with periodic structure, such as homogeneous ones, with holes filled by another material, the $a_{ij}(x)$ is a periodic function of the space variables. When the period of $a_{ij}(x)$ is very small, then u is solution of a certain elliptic problem with constant matrix \tilde{a}_{ij} . In this case matrix \tilde{a}_{ij} is a physical property of the “homogenized” material. Let us now define the function

$$a_{ij}^\varepsilon(x) = a_{ij}(x/\varepsilon)$$
(3)

where ε is a real, positive parameter and let us further consider the following boundary value problem

$$-\frac{\partial}{\partial x_i} (a_{ij}^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_j}) = f(x), \quad \text{in } \Omega$$

$$u^\varepsilon|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega$$
(4)

If $\varepsilon \rightarrow 0$, then $u^\varepsilon \rightarrow \tilde{u}$ weakly in $H_0^1(\Omega)$ and

$$-\frac{\partial}{\partial x_i} (\tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j}) = f(x), \quad \text{in } \Omega$$

$$\tilde{u}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega$$
(5)

where $\tilde{a}_{ij} = \text{const.}$ (see for example [6, p. 56]).

We will discuss below in more detail the asymptotic homogenization approach for one-dimensional, stationary elliptic system with the Dirichlet boundary value conditions.

2. Homogenization of an elliptic system

Let us consider a following one-dimensional (spatial variable $x \in (0, d)$) stationary elliptic boundary value problem with a parameter $\varepsilon > 0$

$$\begin{cases} -\frac{d}{dx} \left(a^\varepsilon(x) \frac{du^\varepsilon}{dx} \right) = f(x), & x \in (0, d) \\ u^\varepsilon(0) = 0 \quad \text{and} \quad u^\varepsilon(d) = 0 \end{cases} \quad (6)$$

where f is a given source term. Equation (6) is, for instance the equation of time-independent heat transfer, equation of electrostatics or magnetostatics. The function u is the temperature, electric potential or magnetic potential respectively and consistently the coefficient $a^\varepsilon(x) > 0$ is the thermal conductivity, dielectric constant or magnetic permeability. Let now $a_0(x)$ be the given basic function determinate on the interval $(0, d)$ called also „the basic cell“. The exemplary basic function a_0 is shown in Figure 1 at $n = 0$. We define the function

$$a^\varepsilon(x) = a_0(x/\varepsilon), \quad x \in (0, d) \quad (7)$$

where ε is a real, positive parameter. Note that the a^ε is a periodic function of period $T = \varepsilon d$. For fixed $\varepsilon > 0$ solution u^ε exists and is unique. It can be shown [1, p. 96], [2, p. 36 and further], that at $\varepsilon \rightarrow 0$ we have $u^\varepsilon \rightarrow \tilde{u}$ (weakly in $H_0^1(0, d)$) and the macroscopic description (homogenized system) constitutes the following elliptic boundary value problem

$$\begin{cases} -\frac{d}{dx} \left(\tilde{a}(x) \frac{d\tilde{u}}{dx} \right) = f, & x \in (0, d) \\ \tilde{u}(0) = 0 \quad \text{and} \quad \tilde{u}(d) = 0 \end{cases} \quad (8)$$

where the effective conductivity coefficient of homogeneous, macroscopic medium is given by the equality [2, p. 44]

$$\tilde{a}(x) = \left[\frac{1}{d} \int_0^d \frac{1}{a_0(x)} dx \right]^{-1} = \left[\frac{1}{\varepsilon d} \int_0^{\varepsilon d} \frac{1}{a^\varepsilon(x)} dx \right]^{-1} = \text{const.} \quad (9)$$

Remark 1. The considered Sobolew space $H_0^1(0, d)$ is definite as follows

$$H_0^1(0, d) = \{u : u \in H^1(0, d), u(0) = 0, u(d) = 0\},$$

$$H^1(0, d) = \{u : u, du/dx \in L^2(0, d)\}.$$

Remark 2. The sequence $u^\varepsilon \rightarrow \tilde{u}$ weakly, when for every function v the convergence

$$\int_0^d a^\varepsilon(x)v(x)dx \rightarrow \int_0^d a(x)v(x)dx$$

is satisfied.

The first integral (from the left) in (9) defines the mean value of the function $1/a_0(x)$ of the basic function, which has a dimension of whole macroscopic medium. The second integral determines the mean value of the function $1/a^\varepsilon(x)$ of the single periodicity cell at optionally chosen $\varepsilon > 0$, too.

From (9) we can easily notice, that the given in advance effective coefficient $\tilde{a} = \text{const.}$ can be obtained at infinitely many forms of basic function $a_0(x)$. If the basic function has a layered structure compose of different ingredients (see for example Fig. 1 at $n = 0$), then the given coefficient $\tilde{a} = \text{const.}$ can be obtained (as a result of homogenization) by mixing the ingredients in various proportions.

An idea presented above of asymptotic homogenization requires that the dimension of the heterogeneity (the dimension of the periodicity cell) will converge to zero. In many applications we meet with the opposite problem, that is the dimension of the periodicity cell is given, while the macroscopic dimension of the sample (the basic cell) grows. The homogenization theory considers such issues using the observation contained in (9) and graduating suitably the microscopic description of the process [2, p. 60]. For example when in (9) we have $\varepsilon = \varepsilon_n = 1/2^n$, $n = 0, 1, 2, \dots$, then the cell dimension grows like 2^n .

3. Approximation of the effective conductivity coefficient

Equality (9) allows us to calculate effective conductivity coefficient $\tilde{a} = \text{const.}$ knowing $a_0(x)$ (or $a^\varepsilon(x)$) determinate on the interval $(0, \varepsilon d)$, at any $\varepsilon > 0$ characterizing inhomogeneous conductivity of the basic cell. The basic function $a_0(x)$ can be approximated for example by the „stepped” function (for simplification denoted by $a_0(x)$ also; see Fig. 2). In such a case our problem becomes a problem of homogenization of layered materials.

Let us pass to more detailed analysis. We discretize the interval $[0, d]$ in the following way: $c_0 = 0 < c_1 < c_2 < \dots < c_m = d$. Further let

$$a_0(x) = \alpha_i > 0 \quad \text{for } x \in (c_{i-1}, c_i), \quad i = 1, 2, \dots, m, \quad c_0 = 0, c_m = d \quad (10)$$

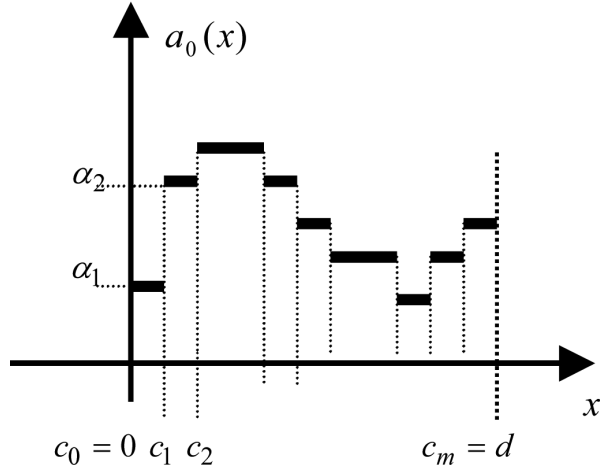


Fig. 2. Stepped basic function $a_0(x)$

Now we normalize the variables (it is not necessary, but can be convenient; see Fig. 5)

$$c_i = \gamma_i d, \quad i = 1, 2, \dots, m, \quad \gamma_0 = 0 < \gamma_1 < \gamma_2 < \dots < \gamma_m = 1, \quad d \in (0, +\infty)$$

$$\alpha_i = \rho_i \beta, \quad i = 1, 2, \dots, m, \quad \rho_i \in (0, 1], \quad \beta = \max_i \alpha_i, \quad \beta \in (0, +\infty)$$
(11)

From (9) as well as (10) and (11) we obtain

$$\tilde{a}(x) = \left[\frac{1}{d} \int_0^d \frac{1}{a_0(x)} dx \right]^{-1} = \beta / \sum_{i=1}^m (\gamma_i - \gamma_{i-1}) / \rho_i, \quad \gamma_0 = 0, \quad \gamma_m = 1$$
(12)

In the case of discretization of the interval $[0, d]$ with the constant step we have $\gamma_i = i/m$ and then

$$\tilde{a}(x) = \frac{m\beta}{\sum_{i=1}^m \frac{1}{\rho_i}}$$
(13)

Let us notice, that (12) describes the function of $2m$ variables $\tilde{a}(\beta, \gamma_1, \dots, \gamma_{m-1}, \rho_1, \dots, \rho_m)$, while (13) describes the function of $m+1$ variables $\tilde{a}(\beta, \rho_1, \dots, \rho_m)$. From this fact appears that the material of given effective conductivity coefficient can be obtained at infinitely many realizations of the “stepped” basic function $a_0(x)$. This observation will be illustrated below with four examples.

4. Examples of numerical calculations

Example 1. Let $m = 2$ and $\rho_2 = 1$ (see Fig. 1 at $n = 0$, and $c_1 = c$, $\alpha_1 = \alpha$). If we take $\gamma_i = \gamma$ as well as $\rho_i = \rho$, then from (12) we obtain (the case considered in the paper [3]) the effective conductivity coefficient in the form:

$$\tilde{a}(\gamma, \rho) = \frac{\rho\beta}{\gamma + (1-\gamma)\rho} \quad (14)$$

Let us try now to determine the family of the “stepped” basic functions at which we obtain the given effective conductivity coefficient \tilde{a} . From (14) we can determine ρ as a function of variable γ and dependent on the parameter \tilde{a} . After elementary calculations we have

$$\rho(\gamma) = \frac{\tilde{a}\gamma}{\beta - \tilde{a}(1-\gamma)} \quad (15)$$

If we want to obtain the material of the given conductivity coefficient \tilde{a} , then any $\gamma \in (0, 1)$ should be taken and then $\rho(\gamma)$ should be calculated from (15). In Figure 3 the function (15) at $\tilde{a} = 0.8$ and $\beta = 2$ is shown. It is easy to notice, that the effective conductivity coefficient $\tilde{a} = 0.8$ of the macroscopic system (after the homogenization process) can be obtained by reading off $\rho(\gamma) \in (0, 0.4)$ from the graph in Figure 3 at any $\gamma \in (0, 1)$. In particular for $\gamma = 0.5$ we have $\rho(0.5) = 0.25$.

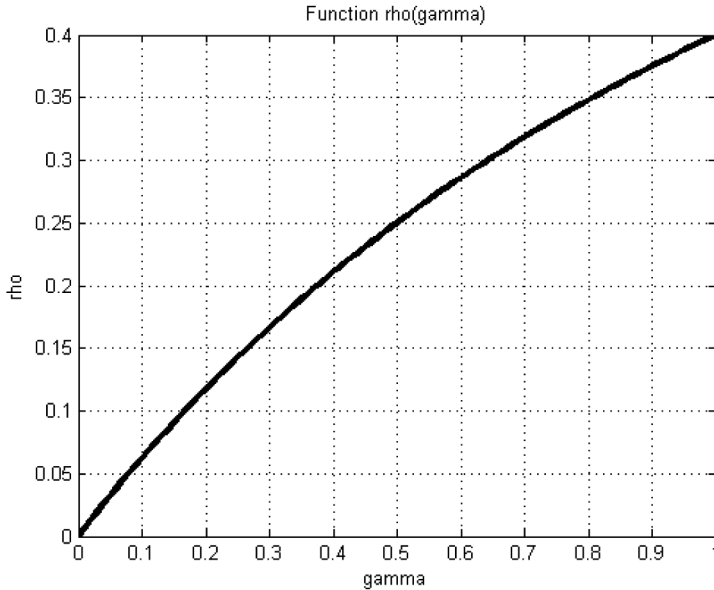


Fig. 3. Function $\rho(\gamma)$ for $\tilde{a} = 0.8$ and $\beta = 2$

The basic function of above considered case for $m = 2$ is shown in Figure 1 (at $n = 0$). Additionally the notation $c = \gamma d$ as well as $\alpha = \rho\beta$ should be taken. In this case the “stepped” approximation of the basic function, in the further homogenization process, can be interpreted as a mixing of two different ingredients in the proportion definite by the parameters γ and ρ .

Example 2. Let us notice, that the coefficient $\tilde{a}(x) = \text{const.}$ given by (12) depends on parameters $\beta, \gamma_1, \gamma_2, \dots, \gamma_{m-1}$ as well as $\rho_1, \rho_2, \dots, \rho_m$, which range is defined in (11). Let $m = 3$, then from (12) we have

$$\tilde{a}(\beta, \gamma_1, \gamma_2, \rho_1, \rho_2, \rho_3) = \frac{\rho_1 \rho_2 \rho_3 \beta}{\rho_2 \rho_3 \gamma_1 + \rho_1 \rho_3 (\gamma_2 - \gamma_1) + \rho_1 \rho_2 (1 - \gamma_2)}, \quad (16)$$

$$0 < \gamma_1 < \gamma_2 < 1, \quad \rho_1, \rho_2, \rho_3 \in (0, 1]$$

The natural question arises: how to select the “stepped” basic function (see Fig. 4) in order to obtain given in advance coefficient $\tilde{a}(x) = \text{const.}$? In the example 1 this problem was solved for $m = 2$ (see equality (15)). Now when $m = 3$ the problem is more complicated, what can be seen in the equality (16). In this case the coefficient $\tilde{a}(x) = \text{const.}$ depends on a few parameters.

Let us determine the effective conductivity coefficient \tilde{a} and the parameters β, ρ_2, ρ_3 . From (16) we have

$$\rho_1(\gamma_1, \gamma_2) = \frac{\tilde{a} \gamma_1}{\beta - \tilde{a} \frac{\gamma_2 - \gamma_1}{\rho_2} - \tilde{a} \frac{1 - \gamma_2}{\rho_3}}, \quad (17)$$

and $0 < \gamma_1 < \gamma_2 < 1$. Similarly if we fix the conductivity coefficient \tilde{a} and the parameters β, ρ_1, ρ_3 , then from (16) we obtain

$$\rho_2(\gamma_1, \gamma_2) = \frac{\tilde{a}(\gamma_2 - \gamma_1)}{\beta - \tilde{a} \frac{\gamma_1}{\rho_1} - \tilde{a} \frac{1 - \gamma_2}{\rho_3}}, \quad (18)$$

For fixed \tilde{a} and β, ρ_1, ρ_2 , from (16) we have

$$\rho_3(\gamma_1, \gamma_2) = \frac{\tilde{a}(1 - \gamma_2)}{\beta - \tilde{a} \frac{\gamma_1}{\rho_1} - \tilde{a} \frac{\gamma_2 - \gamma_1}{\rho_2}}, \quad (19)$$

In Figure 4 the basic function $a_0(x)$ for $m = 3$ and $\rho_3 = 1$ (see (11)) is shown, that is $\alpha_3 = \beta$. In this case to obtain the given in advance effective conductivity coefficient \tilde{a} , the formulas (17) or (18) can be used.

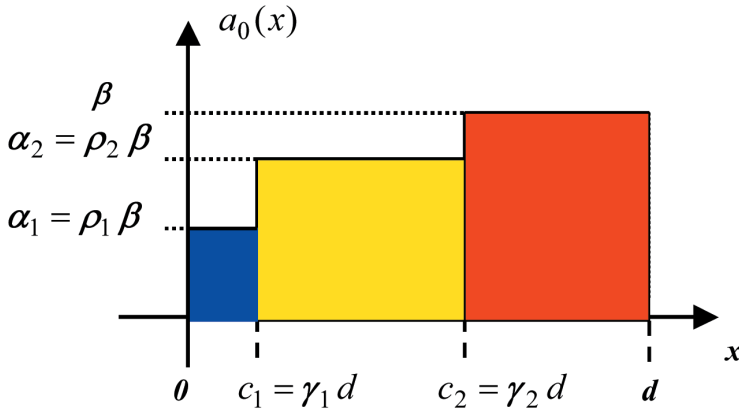


Fig. 4. Stepped basic function $a_0(x)$ for $m = 3$ and $\rho_3 = 1$

Figure 5 presents the function $\rho_1(\gamma_1, \gamma_2)$ definite by (17), and $\tilde{a} = 0.8, \beta = 2$.

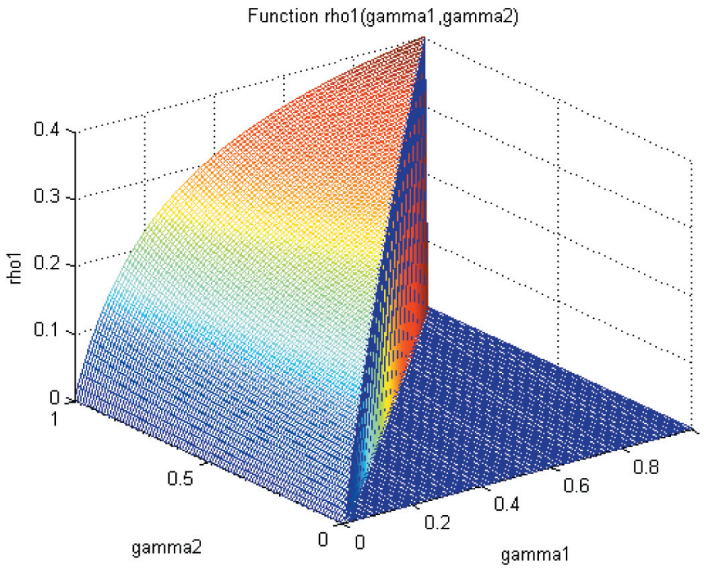


Fig. 5. Function $\rho_1(\gamma_1, \gamma_2)$ for $\tilde{a} = 0.8, \beta = 2$

Figure 6 shows the function (17) for $\tilde{a} = 0.8, \beta = 2$ as well as $\rho_2 = 0.5, \rho_3 = 1$, at three values of $\gamma_2 = 0.2, 0.5, 0.9$.

Let us now consider the case (see Fig. 2) of division of the interval $[0, d]$ into the m equal parts. We have then $\gamma_i = i/m, i = 0, 1, 2, \dots, m$. The effective conductivity coefficient \tilde{a} is in this case defined by (13) and is the function of $m+1$ parameters $\beta, \rho_1, \rho_2, \dots, \rho_m$.

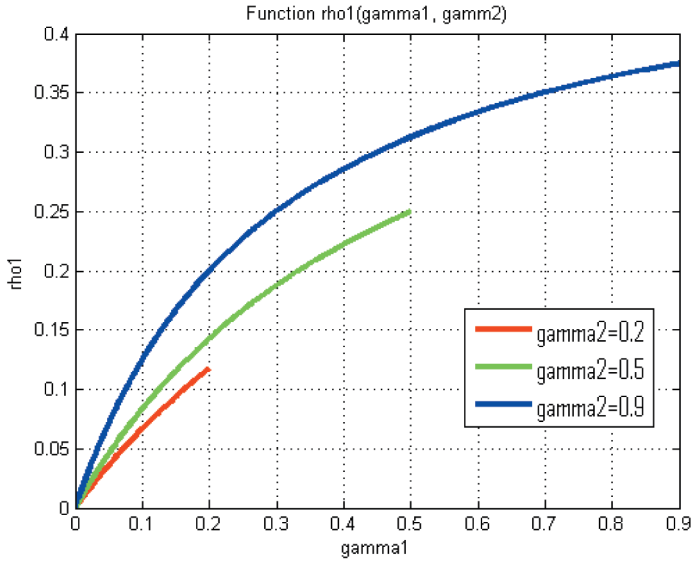


Fig. 6. Function $\rho_1(\gamma_1, \gamma_2)$ for $\tilde{a} = 0.8, \beta = 2$ and $\rho_2 = 0.5, \rho_3 = 1$, at three values of $\gamma_2 = 0.2, 0.5, 0.9$

Example 3. Let $m=2$, then from (13) we obtain

$$\tilde{a}(\rho_1, \rho_2) = \frac{2\beta}{\frac{1}{\rho_1} + \frac{1}{\rho_2}} \tag{20}$$

If like before \tilde{a} is the effective conductivity coefficient given in advance, then from (20) we can calculate that

$$\rho_1(\rho_2) = \frac{\tilde{a}}{2\beta - \frac{\tilde{a}}{\rho_2}}, \quad \rho_2 > \frac{\tilde{a}}{2\beta} \Leftrightarrow \rho_1(\rho_2) > 0 \tag{21}$$

From the condition of physical realizability we have $0 < \rho_i \leq 1$. From inequality (21) at $\rho_2 = 1$ follows that $2\beta > \tilde{a}$, and we know what values of effective conductivity coefficient \tilde{a} can be given in advance at fixed β .

In Figure 7 the function $\rho_1(\rho_2)$ is shown, where function $\rho_1(\rho_2)$ is given by (21) with the following set of parameters: $\tilde{a} = 0.8, \beta = 2$ and $\tilde{a}/2\beta = 0.2 < \rho_2 \leq 1$ (in Fig. 7 for simplification $0.3 < \rho_2 \leq 1$). From (21) we can see that at $\rho_2 \rightarrow 1$ we have $\rho_1(\rho_2) \rightarrow \tilde{a}/(2\beta - \tilde{a}) = 0.25$.

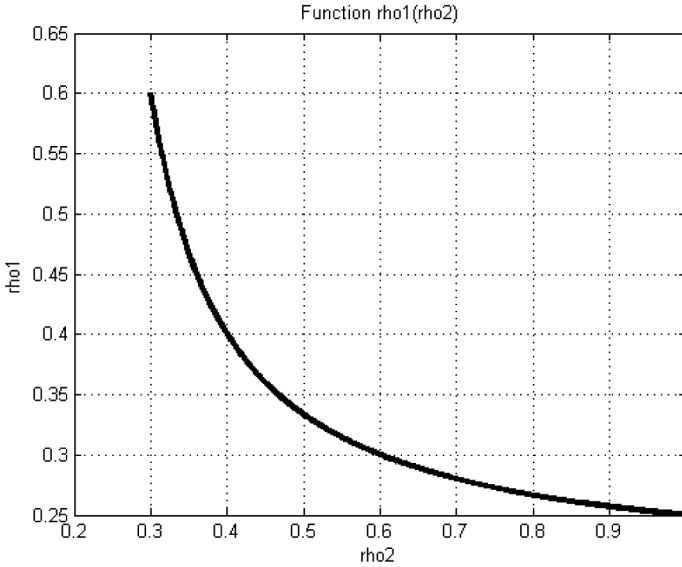


Fig. 7. Function $\rho_1(\rho_2)$ for $\tilde{a} = 0.8$ and $\beta = 2$

Let us notice, that if $\rho_2 = 1$ and $\rho_1 = \rho$ as well as $\gamma_1 = \gamma$, then (the case considered in the example 1, Fig. 1 at $n = 0$) to obtain the value of the effective conductivity coefficient \tilde{a} , the parameter $\rho = \tilde{a}/(2\beta - \tilde{a})$ should be given unequivocally.

Example 4. Let $m = 3$, then from (13) we have

$$\tilde{a}(\rho_1, \rho_2, \rho_3) = \frac{3\beta}{\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}} \quad (22)$$

and from (22)

$$\rho_1(\rho_2, \rho_3) = \frac{\tilde{a}}{3\beta - \tilde{a} \left(\frac{1}{\rho_2} + \frac{1}{\rho_3} \right)}, \quad \frac{3\beta}{\tilde{a}} > \left(\frac{1}{\rho_2} + \frac{1}{\rho_3} \right) \Leftrightarrow \rho_1(\rho_2, \rho_3) > 0 \quad (23)$$

For simplification let us take $\rho_3 = 1$. Then according to (6) $\alpha_3 = \beta$. In consequence the formula (23) becomes the function $\rho_1(\rho_2)$ of one variable ρ_2 . From the condition that $0 < \rho_1(\rho_2)$ we have $\tilde{a}/(3\beta - \tilde{a}) < \rho_2$, while inequality $0 < \rho_2$ implies that $\tilde{a} < 3\beta$, what determines which values of the effective conductivity coefficient \tilde{a} can be taken in advance at fixed β . In Figure 8 the function $\rho_1(\rho_2)$ given by (23) is shown at $\rho_3 = 1$ and the follow-

ing set of parameters $\tilde{a} = 0.8, \beta = 2$ as well as $\tilde{a}/(3\beta - \tilde{a}) < \rho_2 = 0.1538$ (in Fig. 8 for simplification $0.2 < \rho_2 \leq 1$). From (18) at $\rho_3 = 1$ we can see that if $\rho_2 \rightarrow 1$, then $\rho_1(\rho_2) \rightarrow \tilde{a}/(3\beta - 2\tilde{a}) = 0.1818$.

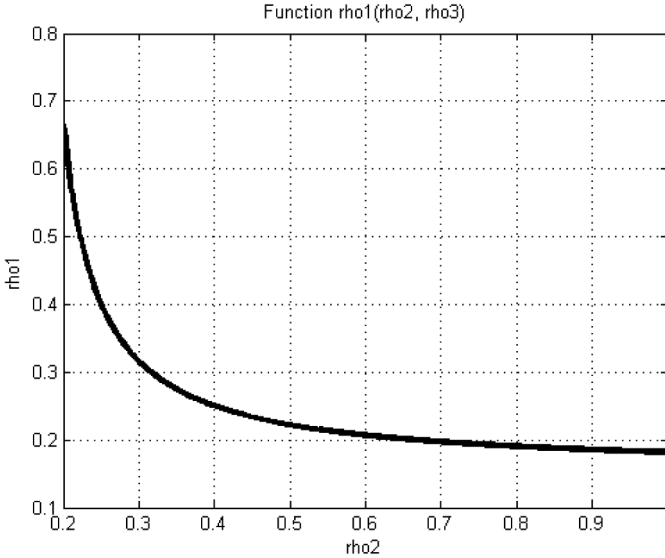


Fig. 8. Function $\rho_1(\rho_2, \rho_3)$ with $\rho_3 = 1$ for $\tilde{a} = 0.8$ and $\beta = 2$

5. Concluding remarks

- The homogenization consists in „mixing” of heterogeneities, characterized by convergence to zero of small parameter $\varepsilon = l/d > 0$, where l is a characteristic dimension of the micro structure and d is a macroscopic dimension. By the heterogeneities we understand e.g. the fibres, micro gaps, etc. in the material structure. The distribution of the heterogeneities can be random or deterministic. There are few approaches to the problem of homogenization [2, p. 16 and further], [7]: asymptotic homogenization, the method of Γ -convergence, the method of the oscillating test functions and some others more. In this paper the idea of the asymptotic homogenization of the media of the periodic microstructure was shown. In the general case the exact determination of the parameters of the macroscopic (homogenized) model is not possible. Only some estimations of these parameters exist.
- In this work the original (in comparison with available papers) method of computing of the effective conductivity coefficient of the homogeneous, macroscopic medium described by the one-dimensional elliptic Dirichlet boundary value problem was presented (see section 3 and examples 2–4). In the calculations the “stepped” approximation

of the basic function was used. Conditions to obtain the given in advance effective conductivity coefficient were shown.

- The following inverse problem is considered too: in what proportions components should be mix, to obtain homogenized material of given properties (see example 1 for $m = 2$ and example 2 for $m = 3$).
- Results of this paper can be generalized to the case of multidimensional spacial variable. For example, for the layered material in R^2 (see Fig. 9) using formulas from the paper [2, p. 99, 109] the matrix $[\tilde{a}_{ij}]$ appearing in (5) can be obtained in the following form

$$\tilde{a} = [\tilde{a}_{ij}] = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix} \quad (24)$$

The matrix \tilde{a} is constant and positively definite with the coefficients given for example in [1, p. 99].

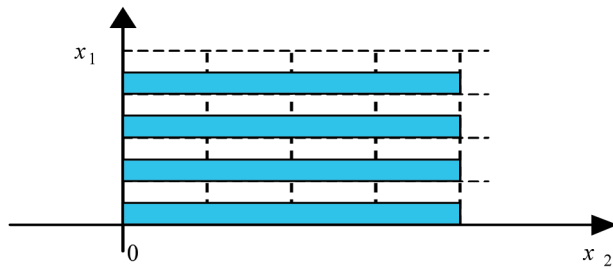


Fig. 9. Layered material

Acknowledgements

Work partially financed from NCN-National Science Centre funds no. N N514644440.

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